Postinstability Models in Dynamics

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ABSTRACT

This paper is devoted to the concept of instability in dynamical systems with the main emphasis on Orbital, Hadamard's and Reynolds' instabilities. It demonstrates that the requirement about differentiability in dynamics in some cases is not consistent with the physical nature of motions, and may lead to unrealistic solutions. Special attention is paid to the fact that instability is not an invariant of motion: it depends upon frames of reference, metric of configuration space, and classes of functions selected for mathematical models of physical phenomena. This leads to the possibility of elimination of certain types of instabilities (and in particular, those which lead to chaos and turbulence) by enlarging the class of functions using the Reynolds-type transformation in combination with the stabilization principle: the additional terms (tile so called Reynolds stresses)are found from the conditions that they suppress the original instability. Based upon these ideas, new approach to chaos and turbulence, as well as new mathematical formalism for nonlinear dynamics, are discussed.

I. INTRODUCTION

In recent years an increasing amount of interest has been addressed to the fact that. in many different domains of science (physics, chemistry, biology, engineering). systems with a similar strange behavior are frequently encountered. These systems display irregular and unpredictable time evolution, and are called chaotic. But chaotic motions are not the only motions in dynamics which are unpredictable. Much earlier. about a hundred years ago, O. Reynolds studied, experimentally and theoretically, turbulent motions in fluids. Despite the many efforts, the problem of prediction of turbulent motions is still unsolved. Later another type of instability which is associated with a failure of hyperbolicity in distributed systems was discovered by J. Hadamard. In all these cases the postinstability behavior of the solutions to the original models is characterized by supersensitivity to initial conditions, and for that reason, it cannot be predicted since the initial Conditions are never known exactly. In this paper we will discuss a possibility to develop a unified approach to prediction of postinstability behavior in dynamics.

1. Mathematical Formulations and Dynamical Invariant

Dynamics describes the motion of systems, i.e. the time evolution of its parameters. The time variable t can be discrete or continuous. In discrete-time dynamical systems, the rate of change of their parameters x is defined only for discrete values of t. These systems can be presented as the iteration of a function:

$$\mathbf{x}_{t+1} = \mathbf{v}(\mathbf{x}_t, t), \quad t = 0, 1, 2, \dots \text{ etc.} \tag{1}$$

i.e. as difference equations.

In continuous-time dynamical systems the rate of change of x is defined for all values of t; such systems can be modelled by ordinary different ial equations.

$$\frac{d\mathbf{x}}{dt} = x = \mathbf{v}(\mathbf{x}, t) \tag{2}$$

or by partial [differential equations:

$$\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}, \mathbf{x}', \mathbf{x}'', \dots t), \quad x' = \frac{\partial x}{\partial s}, \quad x'' = \frac{\partial^2 x}{\partial s^2}, \dots \tag{3}$$

if the rate of change, in addition, depends upon distributions of x over space coordinates s. In Eqs. (l), (2) and (3), x represents the state of the dynamical system.

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Continuous-time dynamical system theory has adopted basic mathematical assumptions of theory of differential equations such as differentiability of the parameters (with respect to time and space) "as many times as necessary," the boundedness of the velocity gradients $\partial \dot{x}/\partial x$ (the Lipschitz conditions) etc. Under these assumptions, the existence, uniqueness and stability of solutions describing the behavior of dynamical systems has been studied. However, the dynamical systems cannot be identified with the mathematical models, i.e. , with the differential equations. Indeed, dynamical systems are characterized by scalars, vectors, or tensors which are invariant with respect to coordinate transformations. Hence, Eqs. (2) or (3) model a dynamical system only if they preserve these invariants after any (smooth) coordinate transformation. For instance, any model "of a mechanical system must be derivable from variational principles which are expressed via the mechanical invariant (kinetic and potential energy, dissipation functions, etc.), In other words, the difference between dynamical systems and the corresponding differential equations is similar to the difference between a matrix as an object of algebra, and a second rank tensor as an object of geometry: The same tensor can be modelled by different matrices depending on choices of coordinates: however, all these matrices must have the same eigenvalues. Continuing this analogy, it can be expected that the parameters x in Eqs. (2), or (3) can be decomposed (at least, in principle) into "invariant" and "non-invariant" components, in the same way in which a matrix A can be decomposed into invariant (diagonal $\tilde{A}$) and coordinate-dependent (t?, θ)[1] components:

$$A = \theta \tilde{A} \theta^{-1} \tag{4}$$

2. **Ignorable** Coordinates and Orbital Instability

In mechanical systems, "non-invariant" components of x can be associated with ignorable (or cyclic) coordinates which do not enter the Lagrangian function explicitly, and therefore, do not affect the energy of the system. For non-conservative systems, in addition to that, the generalized forces corresponding to these coordinates are zero. In terms of Lagrange equations, this property is expressed as the conservation of generalized ignorable impulses P (Gantmacher, F., 1970):

$$\frac{\partial L}{\partial q_\alpha} = 0, \quad Q_\alpha = 0, i.e. \quad \frac{\partial L}{\partial \dot{q}_\alpha} = P_\alpha = Const, \alpha = 1, 2, \dots m \tag{5}$$

unlike the equations for the position coordinates which, in general, do not preserve the position impulses P:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = Q_k, \quad k = 1, 2, \dots n \tag{6}$$

Here L is the Lagrangian, q_α and q_k are ignorable and position coordinates, respectively. Q_k are non-potential components of generalized forces.

In order to illustrate the difference between position and ignorable coordinates, consider the following dynamical system:

$$\dot{r} = \sin r, \quad \dot{\theta} = \omega = Const \qquad (7)$$

where r and θ are polar coordinates.

It has periodic attractors:

$$r = \frac{\pi k}{2}, \quad k = 0, 1, \dots etc., \quad \theta = \theta_o + \omega t \qquad (s)$$

Returning to Eq. (7), one can easily identify r and θ as position and ignorable coordinates, respectively, Indeed, the Lagrangian and generalized forces for this dynamical system are:

$$L = \frac{1}{2}(\dot{r}^2 + \dot{\Theta}^2), Q_r = \dot{r}\cos r, Q_\theta = 0 \qquad (9)$$

and therefore.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_r} = Q_r \neq 0, \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\theta} = Q_\theta = 0 \qquad (10)$$

It is important to emphasize that the position coordinate r is stable at the attractors, while the ignorable coordinate Θ is at the boundary of stability: any small error in Θ will increase linearly (but not exponentially) in time.

Indifference of energy of a dynamical system to an unlimited growth of ignorable coordinates raises the following question: do there exist such states where all the position coordinates are stable, but some of ignorable coordinates are unstable? Numerical experiments give positive answers to this question. These states are associated with chaotic behavior. Unlike periodic attractors, here any small error in initial values of ignorable coordinates increases exponentially (but not linearly) with time, so that two motion trajectories which initially were indistinguishable (because of finite scale of observation), diverge exponentially, and therefore, a behavior of the dynamical system becomes unpredictable. But is such a "multivaluedness" of trajectories consistent with the basic mathematical assumptions about motions of dynamical systems? This problem will be discussed in the next sections in connection with predictability in classical dynamics.

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3. Distributed Systems and Failure of differentiability

There are two types of distributed systems - hyperbolic and parabolic - which can model dynamical behaviors. (Elliptic equations are ill-posed for time evolution processes.) Distributed dynamical systems can exhibit more sophisticated behaviors such as turbulence (whose relation to chaos is still disputed), Hadamard's instability (Zak, M., 1982a, b, c) which is associated with failure of hyprebolicity and transition to ellipticity, formation of cumulative effects (Zak, M., 1970, 19S3), etc.

Actually, all these phenomena are associated with spatial effects in distributed dynamical systems resulting from additional mathematical restrictions requiring differentiability of dynamical parameters with respect to spatial coordinates. But are these restriction always consistent with physical nature of motions'? The following example shows that such restrictions may lead to unrealistic solutions.

Consider an ideal filament stretched in a vertical direction, as shown in Fig. 1. Let us crosscut it in a middle point and observe the behavior of upper and below parts. The below part will be folding up in a "thick point", losing differentiability of its configuration. The upper part will preserve differentiability of its configuration in an open interval (which does not include the free end), but at the end small initial disturbances will accumulate and become infinitely large (snap of a whip). Both of these effects are lost in the dynamical model based upon differentiability of the dynamical parameters (for the below part of the filament) and upon the Lipschitz condition at the free end (for the upper part of the filament (Zak, M., 1970).

4. Open Problems

As has been illustrated below, the evolution of ignorable coordinates may be fundamentally different from the evolution of non-ignorable (or position) coordinates. For instance, the growth of position coordinates is limited by the boundedness of the system energy, and consequently, their instability cannot persist: the system must find an alternative stable state. In contradistinction to that, the instability of ignorable coordinates (which is called an orbital instability) can persis all the time without having an alternative stable state. In particular, the indifference of the energy to changes of ignorable parameters is responsible for such phenomena as turbulence, chaos, failure of differentiability and

uniqueness of solutions. In turn, the occurrence of these phenomena questions the basic mathematical assumptions about the claw of functions in whit-h the dynamical systems are described.

The existence of two different types of parameters in dynamical systems raises some other questions: can instability of ignorable coordinate develop independently on the behavior of the position coordinates? Is instability of ignorable coordinates an invariant of the frame of reference, or of the class of functions in which motions are studied? Can instability of ignorable coordinate be eliminated by change of motion representation?

The answers to these questions, as well as new representations of chaos and turbulence will be discussed in this paper.

II.   INSTABILITY IN DYNAMICS

1.   Basic Concepts

Most of the dynamical processes are so complex that their universal theory which would capture all the details during all the time periods is unthinkable. That is why the art of mathematical modelling is to extract only the fundamental aspects of the process and to neglect its insignificant features, without losing the core of information. But "insignificant features" is not a simple concept,. In many cases even vanishingly small forces can cause large changes in t he dynamical system parameters, and such situations are intuitively associated with the concept of the instability. Obviously the destabilizing forces cannot be considered as "insignificant, features", and therefore, they cannot be ignored. But since they may be humanly indistinguishable, in the very beginning, there is no way to incorporate them into the model. This simply means that the model is not adequate for quantitative description of the corresponding dynamical process: it must be changed or modified. However, the instability delivers an important qualitative information: it manifests the boundaries of applicabili t y of t he original model.

We will distinguish short and long-term instabilities. Short-term instability occurs when the system has alternative stable states. For (dissipative systems such states can be represented by static or periodic attractors. In the very beginning of the post-instability transition period, the unstable motion cannot be traced quantitatively, but it becomes more

and more deterministic as it approaches the attractor. Hence, a short-term instability does not necessarily require a model modification. Usually this type of instability is associated with bounded deviation of position coordinates whose changes affect the energy of the system. Indeed, if the growth of a position coordinate persists, the energy of the system would become 11111)01111 (1('(1.

The long term instability occurs when tile system does not have an alternative stable state. Such type of instability can be associated only with ignorable coordinates since these coordinates do not effect the energy of the system. The long term instability will be the main subject of this paper.

2. Orbital instability

a. *Ignorable Coordinates*

As mentioned the Intro duction (see (5)), the coordinate q_α is called ignorable if it does not enter the Lagrangian function L as well as nonconservative generalized forces Q:

$$\frac{\partial L}{\partial q_\alpha} = 0, \quad Q_\alpha = 0 \tag{11}$$

therefore,

$$\frac{\partial L}{\partial \dot{q}_\alpha} = P_\alpha = Const \tag{12}$$

i.e., the generalized ignorable impulse P_α is constant.

As follows from Eq.(12), there exist such states of dynamical systems (called station-ary motions) that all the position coordinates retain constant value while the ignorable coordinates vary in accordance with a linear law. For example, a regular precession of a heavy symmetric gyroscope. is a stationary motion characterized by the equation:

$$\Theta = Const, \quad \dot{\psi} = Const, \quad \dot{\phi} = Const \tag{13}$$

where the angle of precession ψ and the angle of pure rotation ϕ are ignorable coordinates, while the angle of nutation Θ - an angle formed by the axis of gyroscope and the vertical is a position coordinate.

Obviously, stationary motions are not stable with respect to ignorable velocities: a small change in $\dot{q}_\alpha$ at t = O yields, as time progresses. an arbitrarily large change in the ignorable coordinates themselves. However, since this change increases linearly (but not exponentially), the motion is still considered as predictable. In particular, the Lyapunov exponents for stationary motions are zero:

$$\sigma = \lim_{d(0)\to 0 \, , \, t\to\infty} \left(\frac{1}{t}\right) \ln \frac{d(0)t}{d(o)} = 0 \qquad (14)$$

However, in case of nonstationary motions, tile ignorable coordinate can exhibit more sophisticated behaviors. In" order to demonstrate this, let us consider an inertial motion of a particle M of unit mass on a smooth pseudosphere S having a constant negative curvature (Fig. 2):

$$G_o = Coil.st < 0 \qquad (15)$$

Remembering that trajectories of inertial motions must be geodesics of S, we will compare two different trajectories assuming that initially they are parallel and that the distance between them, ϵ_0, is very small.

As shown in differential geometry, the distance, between such geodesics will exponentially increase:

$$\epsilon = \epsilon_0 e^{\sqrt{-G_0}\,t}, \; GO < 0 \qquad (16)$$

Hence, no matter how small the initial distance ϵ_0, the current distance ϵ tends to infinity.

Let us assume now that the. accuracy to which the initial conditions are known is characterized by L. It means that any two trajectories cannot be distinguished if the distance between them is less than L, i.e. if:

$$\epsilon < L \qquad (17)$$

The period during which the inequality (17), holds has the order:

$$\Delta t \sim \frac{1}{|\sqrt{-G_0}|} \, ln \frac{L}{\epsilon_0} \qquad (18)$$

However. for

$$t >> \Delta t \qquad (19)$$

these two trajectories diverge such that they can be distinguished and must be considered as two different trajectories, Moreover, the distance between them tends to infinity even if ε_0 is small (but not infinitesimal). That is why the motion, once recorded, cannot be reproduced again (unless the initial conditions are known exactly), and consequently, it attains stochastic features. The Lyapunov exponent for this motion is positive and constant:

$$\sigma = \lim_{\substack{t \to \infty \\ d(0 \to 0}} \left(\frac{1}{t}\right) ln \frac{\epsilon_0 e^{\sqrt{-G_0}t}}{\epsilon_0} = \sqrt{-G_0} = \text{Const} > 0 \tag{20}$$

Let us introduce a system of coordinates at the surface S: the coordinate q_1 along the geodesic meridians, and the coordinate q_2 along the parallels. In differential geometry such a system is called semi-geodesical. Tile square of the distance between adjacent point on the pseudosphere is:

$$ds^2 = g_{11} dq_1^2 + 2g_{12}dq_1 dq_2 + g_{22}dq_2^2 \tag{21}$$

where

$$g_{11} = 1, \quad g_{12} = 0, \quad g_{22} = -\frac{1}{G_0}e^{-2\sqrt{-G_0}q_1} \tag{22}$$

The Lagrangian for the inertial motion of the particle M on the pseudosphere is expressed via the coordinates and their temporal derivates as:

$$L = g_{ij}\dot{q}_i\dot{q}_j = \dot{q}_1^2 - \frac{1}{G_0}e^{-2\sqrt{-G_0}q_1}\dot{q}_2^2 \tag{?3}$$

anti, consequent 1 y,

$$\frac{\partial L}{\partial q_2} = 0 \tag{24}$$

while

$$\frac{\partial L}{\partial q_1} \neq 0, \text{if } \dot{q}_2 \neq 0 \tag{25}$$

Hence, q_1 and q_2 play roles of position and ignorable coordinates, respectively.

Therefore, an inertial motion of a particle on a pseudosphere is stable with respect to the position coordinate q_1, but it is unstable with respect to the ignorable coordinate. However, in contradistinction to the stationary motions considered above, here the instability is characterized by exponential growth of the ignorable coordinate, and that is wily the motion becomes unpredictable. It can be shown that such a motion becomes stochastic, (Arnold, V., 1988).

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Instability with respect to ignorable coordinates can be associated with orbital instability. Indeed, turning to the last example, one can represent the particle velocity v as the product:

$$\mathbf{v} = |v|\tau \tag{26}$$

In the course of the instability, tile velocity magnitude $|\mathbf{v}|$, and consequently, the total energy, remain unchanged, while all tile changes affect only τ, i.e. the direction of motion, In other words, orbital instability leads to redistribution of the total energy between the coordinates, and it is characterized by positive Lyapunov exponents.

b. Orbital Instability of Inertial Motions

The results described above were related to inertial motions of a particle on a smooth surface. However, they can be easily generalized to motions of any finite-degree-of-freedom mechanical system by using the concept of configuration space. Indeed, if tile mechanical system has N generalized coordinates q^1 ($2' = 1, 2, \ldots . N$) and is characterized by the kinetic energy:

$$W = \alpha_{ij}\dot{q}^i\dot{q}^j \tag{27}$$

then the configuration space can be introduced as an N-dimensional space with the following metric tensor:

$$g_{ij} = \alpha_{ij} \tag{2s}$$

while the motion of the. system is represented by the motion of the unit-mass particle in this configuration space.

In order to continue the analogy to the motion of the particle on a surface in actual space we will consider only two-dimensional subspaces of the N-dimensional configuration space, without loss of generality. Indeed, a motion which is instable in any such subspace, has to be qualified as *an* unstable in the entire configuration space.

Now the Gaussian curvature of a two-dimensional configuration subspace (q^1, q^2) follows from the Gauss formula:

$$G = \frac{1}{a_{11}a_{22} - a_{12}^2}\left(\frac{\partial^2 a_{12}}{\partial q^1 \partial q^2} - \frac{1}{2}\frac{\partial^2 a_{11}}{\partial q^2 \partial q^2} - \frac{1}{2}\frac{\partial^2 a_{22}}{\partial q^1 \partial q^1}\right)$$
$$- \Gamma^\gamma_{12}\Gamma^\delta_{12}\, a_{\gamma\delta} - \Gamma^\alpha_{11}\Gamma^\beta_{22}a_{\alpha\beta} \tag{29}$$

where the connection coefficients Γ^l_{sk} are expressed via the Christoffel symbols:

$$\Gamma^l_{sk} = \frac{1}{2} a^{lp} \left(\frac{\partial a_{sp}}{\partial q^k} + \frac{\partial a_{kp}}{\partial q^s} - \frac{\partial a_{sk}}{\partial q^p} \right) \qquad (30)$$

while

$$a^{\alpha\beta} a_{\beta\gamma} = a^\alpha_\gamma = \begin{cases} 0 & \text{if } \alpha \neq \gamma \\ 1 & \text{if } \alpha = \gamma. \end{cases} \qquad (31)$$

Thus, the Gaussian curvature of these subspaces depends only on the coefficients a_{ij}, i.e. it is fully determined by the kinematical structure of the system [see equation (27)]. In case of inertial motions, the trajectories of the representative particle must be geodesics of the configuration space. Indeed, as follows from (26):

$$\frac{d\tau}{dt} = \frac{d\tau}{ds} \dot{s} = 0 \text{ if } \dot{v} = 0, \text{ and } |v| = |\dot{s}| = Const \neq 0 \qquad (32)$$

where s is the arc coordinate along the particle trajectory:

$$ds = a_{ij} dq^i dq^j \qquad (33)$$

But then:

$$\frac{d\tau}{ds} = 0 \qquad (34)$$

which is the condition that the trajectory is geodesic.

If the Gaussian curvature. (29) which is uniquely defined by the parameters of the dynamical system a_{ij}, is negative:

$$G < 0 \qquad (35)$$

then the trajectories of inertial motions of the system originated at close, hut different points of the configuration space diverge exponentially from each other, and the motion becomes unpredictable and stochastic. Some examples of orbital instability in inertial motions are discussed by Zak, M. 1985b.

c. Orbital Imitability of Potential Motions

Turning back to the motion of the particle M on a smooth pseudosphere (Fig. 2), let us depart from inertia] motions and introduce a force F acting on this particle. For noninertial motions (F # O) the trajectories of the particle will not be geodesics, while the rate of their deviation from geodesics is characterized by the geodesic curvature χ. It is obvious that this curvature must depend on t he forces F:

$$\chi = \chi(F) \qquad (36)$$

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L, Synge (1926 has shown that if the force F is potential:

$$F = - \nabla \Pi \tag{37}$$

where Π is the potential energy, then the condition (35) is replaced by the following:

$$G_0 \; + \; 3\chi^2 W + \left(\frac{1^2 \Pi}{\partial q^i \partial q^j} - \Gamma_c^k \right) \frac{\partial \Pi}{\partial q^k} n^i n^j < 0; \; i, j = 1, 2 \tag{3S}$$

Here Γ_{ij}^k are defined by Eqs. (30), and n^i are the contravariant components of the unit normal n to tile trajectory.

The geodesical curvature χ in (38) can be expresser via the potential force F:

$$\chi = \frac{\mathbf{F} \cdot \mathbf{n}}{2W} = - \frac{\nabla \Pi \cdot \mathbf{n}}{2W} \tag{39}$$

As follows from (3S) and (39), the condition (38) reduces to (35) if F = O.

Suppose for example, that the following elastic force:

$$F = -\alpha^2 \epsilon, \alpha^2 = Const \tag{40}$$

proportional to the normal deviation ϵ from the geodesic trajectory is applied to the particle M moving on the smooth pseudosphere. If the initial velocity is directed along one of the meridians (which are all geodesics), the unperturbed motion will be inertial, and its trajectory will coincide with this meridian since there $\epsilon = O$, and therefore, F = 0. In order to verify the orbital stability of this motion, let us turn to the criterion (3S). Since:

$$\chi = 0, \; \text{and} \; \frac{\partial \Pi}{\partial q^k} = F^k = 0 \tag{41}$$

for the unperturbed motion, one obtains the condition for orbital stability:

$$G_0 + \frac{\alpha^2}{2W} > 0, \; \text{i.e.} \; \alpha^2 < -2WG, \; G < 0 \tag{42}$$

where

$$W = \frac{1}{2} m v_0^2 \tag{43}$$

As in the case of inertial motions, tile inequality:

$$\alpha^2 < -2WG_0 \tag{-M}$$

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leads to unpredictable (stochastic) motions which are characterized by:

$$\sigma = \sqrt{G_0 - \frac{\alpha^2}{2W}} = \text{Const} > 0 \qquad (45)$$

For pure inertial motions ($\alpha = 0$), Eq. (45) reduces to Eq. (15).

After the discovery of chaotic attractors, the stochastic motions which are generated by the instability and are characterized by positive Lyapunov exponents, are called chaotic. Hence, the inequalities (35) and (50) can be associated with criteria of chaos: if the left hand part in (50) is bounded away from zero by a negative number -B in all the configuration space where the motion can occur, then the motion will be chaotic, and its positive Lyapunov exponent will be:

$$\sigma \geq B^2 \qquad (46)$$

Unfortunately, this criterion is too "strong" to be of practical significance: it is sufficient. but not necessary. Indeed, this criterion assumes that not only global, but also the local Lyapunov exponents are positive in any point of the configuration space. At the same time, for many chaotic motions, local Lyapunov exponents in certain domains of the configuration space are all negative, or zero, although some of the global exponents are still positive.

d.   *General Case*

Following J. L. Synge, the results for the orbital instabilit y of inertial and potential motions for a system of material points can be generalized to arbitrary motions.

Since the motion of a system of material points in the configuration space with the metric (28) is represented by a unit-maw point, the momentum equation follows from tile second Newton's law:

$$q'' + \Gamma^r_{mn} \dot{q}^m \dot{q}^n = Q' \qquad (47)$$

where Q'' is the force applied to the point. Let q' be the coordinates of the representative point M moving along an undisturbed natural trajectory C, and $(q^r + \eta^r)$ the coordinates of the corresponding (simultaneous) point M^* of the disturbed natural trajectory C, while η^r is an infinitesimal disturbance vector. The condition for stability of the motion is that the magnitude of the disturbance vector should remain permanently small,

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introducing a unit disturbance vector μ^r co-directional with η^r, so that:

$$\eta^r = \eta\mu^r, \quad a_{mn}\,\mu^m\mu^n = 1 \tag{48}$$

where η is the disturbance vector magnitude, and substituting $q^r + \eta^r$ into Eq. (47), one can obtain an invariant differential equation with respect to the scalar η (Synge, 1926):

$$\ddot{\eta} + A\eta = 0 \tag{49}$$

where the scalar A is uniquely defined by the metric coefficients a_{ij} and the forces Q_i, namely:

$$A = G_{mnsl}\mu^m\dot{q}^n\mu^s\dot{q}^l - \dot{\mu}^2 - Q_{rs}\mu^r\mu^s \tag{50}$$

in which G_{mnsl} is the curvature tensor of the configuration space expressed in the covarian form:

$$G_{msnl} = \frac{\partial\Gamma_{nl}^m}{\partial q^s} - \frac{\partial\Gamma_{ns}^m}{\partial q^l} + \left(\Gamma_{nl}^u\Gamma_{ms}^v - \Gamma_{ns}^u\Gamma_{ml}^v\right)\alpha^{uv}, \quad a^u\,a_{vp} = \delta_u^p \tag{51}$$

Here Γ_{mn}^r are the Christoffel symbols defined by Eq. (30), and

$$Q_{rs} \cdot \left(\frac{\partial Q^r}{\partial q^l} + \Gamma_{ln}^r Q^n\right)a_{ls}, \quad \tilde{\mu}^r = \Gamma_{mn}^r\mu^m\dot{q}^n, \tilde{\mu} = |\tilde{\mu}^r| \tag{52}$$

while the metric tensor of the configuration space is given by Eq.(27).

Eq. (49) leads to a sufficient condition for a dynamical system given in the form (47) to be exponentially unstable. If the Riemannian curvature of the manifold of configurat ions corresponding to every t we-space element $x^m x^n$ containing the direction of the given trajectory is bounded away from zero by a constant negative value, and $Q_{mn}x^m x^n$ is bounded away from zero by a constant positive value in all the domains of the configuration space where the motion can occur, then the motion will be exponentially unstable: since this instability persists, the motion will attain stochastic features (as in the case of the inertial or potential motion of a particle on a smooth pseudosphere), and therefore, it will become chaotic. Actually the condition *(38)* which was formulated earlier without a proof, follows directly from Eq. (49).

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Obviously, the persistency of tile instability in Eq. (49) can occur only due to a contribution of the exponential growth of the ignorable coordinates into the total magnitude of the disturbance vector η. For instance, in the case of inertial motion of the particle M on a smooth p seudosphere, tile disturbance vector can be represented by tilt' components ϵ and ν which are co-directional and normal to t he IIIll)t'l'till')t'[l (geodesic) trajectory. The component ν cor responds to the ignorable coordinate, and its evolution is described by Eq.(49) which reduces to:

$$\ddot{\nu} + 2W_0 G_0 \nu = 0 \tag{53}$$

The exponential instability of ν when $G_0 = \text{Const} < 0$ leads to chaos. At the same time, the position coordinate ϵ is eliminated from Eq.(53) and it can be found from the energy conservation:

$$\epsilon = \epsilon_0 \frac{\dot{\epsilon}}{\dot{S}_0} \tag{54}$$

where $\dot{S}_0$ and ϵ_0 are the initial conditions at t = O for the motion velocity along the trajectory, and the position coordinate of the disturbance vector, respectively. In spite of some limitations of the results described above (the conditions for chaos are sufficient, but not necessary, the forces Q^r depend only upon coordinates, but not upon velocities), they nevertheless elucidate physical origin of orbital instability, chaos, and consequently, of unpredictability of motions in classical dynamics.

3. Hadamard's Instability

a. *General Remarks*

The results presented in the previous section can be applied to distributed systems after a discretization technique which reduces them to a finite-dimensional systems. For instance, as noticed by **Arnold** (1988), an inviscid stationary flow with a smooth velocity field:

$$v_x = A \sin Z + C \cos Y, \; v_y = B \sin X + A \cos Z, \; v_z = C \sin Y + B \cos X \tag{55}$$

has chaotic trajectories X(t), Y(t), Z(t) of fluid particles (**Lagrangian** turbulence) due to negative curvature of the configuration space which is obtained as a finite-dimensional approximation of a continuum. However, there are some special types of instability in distributed systems which can be lost in the course of the discretization, and they will be focused in this section.

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As noticed in the previous section, the long-term instability which may lead to chaos, is associated with the orbital instability, i.e. with the instability of ignorable coordinates. However, in distributed systems described by partial differential equations, there is another possibility for long-term instability which is associated with the decrease of scale of motions, i.e. with the growth of spatial derivates of the system parameters. In mathematical terms it nl('ails a failure of differentiability of the solutions to the corresponding governing equations. However, an unlimited growth of spatial derivates must be consistent with the boundedness of energy. Indeed, the stresses in continuous media depend not upon displacements or velocities. but upon their gradients, i.e. upon their space derivatives. Hence, We have to find such situations when an unlimited growth of these derivative does not lead to unbounded stresses.

Turning to geometry of displacements and their gradients in continua, let us introduce the displacement vector:

$$u = r - r_o \tag{56}$$

where r_o and r are the radii-vectors of the same particle before and after deformation, respectively. In elastic bodies, the stress tensor depends upon the displacement gradien ∇u via the strain tensor ϵ:

$$\epsilon = \frac{1}{2}\left[\nabla u + (\nabla u)^T + u \cdot (\nabla u)^T\right] = \frac{1}{2}\left[\nabla r \cdot (\nabla r)^T - \overset{o}{g}\right] \tag{57}$$

where $\overset{o}{g}$ is the unit (the. initial state) tensor, while the current state metric tensor is defined as:

$$g = 2\epsilon + \overset{o}{g} \tag{58}$$

The tensor-gradient ∇r in (57) can be decomposed as:

$$\nabla r = CB \tag{59}$$

where C is a symmetric tensor:

$$C = +\left[\nabla r \cdot (\nabla r)\right]^{1/2} \tag{60}$$

and B is an orthogonal tensor:

$$B = +\left[\nabla r \cdot (\nabla r)^T\right]^{1/2} \cdot (\nabla r)^T = (B^{-'}), \quad \det B = 1 \tag{61}$$

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.4s follows from (60), the strain tensor

$$\epsilon = \frac{1}{2}(c^2 - \overset{\circ}{g})$$ (62)

and consequently, the stress tensor depends only upon tile symmetric part of the tensor-gradient ∇ r , and does not depend upon its orthogonal component B which corresponds to rigid rotations of elementary volumes. However, indirectly an unlimited growth of these r o t a t i o n s can lead to unbounded stresses in three-dimensional elastic bodies. Indeed. as follows from the identity:

$$\nabla \times \nabla \mathbf{r} = 0$$ (63)

the components of the tensor-gradient ∇ r must satisfy six additional constraints which are called the compatibility equations. Loosely speaking, they follow from the require-ment that after deformations the continuum should not have any "holes" or "cracks". In geometrical terms, Eqs. (62) represent the fact that after deformations, the actual space remain Euclidean, i.e. the curvature tensor is zero:

$$R = 0$$ (64)

However, six constraints imposed upon the tensor-gradient ∇ r by Eqs. (63), or (64) are also not independent. Indeed, according to another identity:

$$\nabla \cdot \nabla \times \nabla \mathbf{r} \equiv 0$$ (65)

which holds even if

$$\nabla \times \nabla \mathbf{r} \neq 0$$ (66)

and which is equivalent to three scalar equations, only three of the six constraints (63) are true independent. In geometrical terms, Eq. (65) can be associated with the Bianchi identities, (W. Fluge, 1962).

Thus, nine components of the vector-gradient ∇ r must satisfy three independent compatibility equations, and therefore, if all six components of the stress tensor ϵ are given, then the remaining three components of ∇ r and consequently, all the rigid rotations. will be uniquely defined. This means that in isotropic three-dimensional elastic bodies, an unlimited decrease of scale of motions would lead to unbounded stresses which is physically impossible.

Let us turn to one-dimensional continua (filaments). In this case, rigid rotations define the external geometry of the model (the rotations about the binormal to the filament correspond to tile first curvature, and the rotations about the tangent to the filament correspond to the second curvature, or twist, Fig. 3), and they do not depend upon the elongations of tile curve which define the stress. Indeed, let us introduce the filament equation in the form:

$$\mathbf{r} = \mathbf{r}(\psi, t), \left|\frac{\partial \mathbf{r}}{\partial \psi}\right| = 1 \tag{67}$$

where ψ plays the role of an Eulerian coordinate. Then the motions associated with changes of the internal geometry, and therefore, the stresses are described by the function:

$$\psi = \psi(s, t) \tag{68}$$

where s is a Lagrangian coordinate of individual particle.

At the same time, the curvatures of the filament configurations can be expressed as:

$$\Omega_3 = \left|\frac{\partial^2 \mathbf{r}}{\partial \psi^2}\right|, \quad \Omega_1 = \frac{\left(\frac{\partial \mathbf{r}}{\partial \psi} \times \frac{\partial^2 \mathbf{r}}{\partial \psi^2}\right) \cdot \frac{\partial^3 \mathbf{r}}{\partial \psi^3}}{\left|\frac{\partial^2 \mathbf{r}}{\partial \psi^2}\right|} \tag{69}$$

Consequently, both curvatures are independent upon the internal geometry characterized by (68), and in particular, upon the stress defined by the derivative $\partial \psi / \partial s$.

This means that unlimited growth of the curvature may not cause stress at all, and therefore, the instability in the form of unlimited decrease of scale of motions is possible (Fig. 3b).

The situation becomes more complicated in two-dimensional continua (films, membranes). Here the internal geometry is defined by two-dimensional versions of Eqs. (56)-(63), while the external geometry is described by the coefficients of the second fundamental form:

$$b_{ij} = \frac{\partial^2 \mathbf{r}}{\partial \psi^i \partial \psi^j} \cdot \mathbf{n}, \quad i, j, = 1, 2 \tag{70}$$

where ψ^i are coordinates on the surface, and n is the unit normal to the surface.

However, these coefficients are not independent: they are coupled with the strains by three compatibility equations:

$$b_{11}b_{22} - b_{12}^2 = \Gamma_{12}^v \Gamma_{12}^\delta g_{v\delta} - \Gamma_{11}^\alpha \Gamma_{22}^\beta g_{\alpha\beta} -$$

$$\frac{1}{2}\frac{\partial^2 g_{11}}{\partial \psi^{(2)2}} \;+\; \frac{\partial^2 g_{12}}{\partial \psi^{(1)}\partial \psi^{(2)}} \;-\; \frac{1}{2}\frac{\partial^2 g_{22}}{\partial \psi^{(1)2}}, (\nu, \delta, \alpha, \beta = 1, 2) \tag{71}$$

$$\frac{\partial b_{il}}{\partial \psi^{(2)}} \;-\; \frac{\partial b_{i2}}{\partial \psi^{(1)}} \;=\; \Gamma^1_{i2}b^1_{11} - \Gamma^2_{i1}b_{22} + (\Gamma^2_{i2} - \Gamma^1_{i1})b_{12}, (i = 1, 2) \tag{72}$$

where

$$g_{11} = 1 + 2\epsilon_{11}, g_{12} = 2\epsilon_{12}, \quad g_{22} = \overset{0}{g}_{22} + 2\epsilon_{22} \tag{73}$$

The two-dimensional Christoffel symbols are:

$$\Gamma^n_{ij} = \frac{1}{2}g^{nl}\left(\frac{\partial g_{li}}{\partial \psi^{(j)}} + \frac{\partial g_{lj}}{\partial \psi^{(i)}} - \frac{\partial g_{ij}}{\partial \psi^{(l)}}\right), (n, i, j = 1, 2) \tag{74}$$

while

$$\|g_{ij}\| = \|g^{ij}\|^{-1} \tag{75}$$

Hence, in general, three coefficients b_{ij} are defined by the strains ϵ_{ij} from the three equations (71) and (72), and consequently, change in b_{ij} affects the strains ϵ_{ij}.

Nevertheless, there are situations when the unlimited growth of curvature may not effect the stress at all. In order to describe this case, recall that on a surface with negative *or* zero Gaussian curvature

$$G = \frac{b_{11}b_{22} - b_{12}}{g_{11}g_{22} - g^2_{12}} \tag{76}$$

there exists a family of asymptotical lines where the second fundamental form is equal to zero:

$$b_{11}\tan^2\phi \;+\; 2b_{12}\tan\phi \;+\; b_{22} = 0 \tag{77}$$

while the **angle ϕ between** an asymptotical line and the coordinate line ψ_1 is found as:

$$\tan\phi = -\frac{b_{12}}{b_{11}} \pm \frac{\sqrt{b^2_{11} - b_{11}b_{22}}}{b_{11}}, (b_{11} \neq 0) \tag{78}$$

Selecting the **coordinate ψ_1 as** an asymptotic line, one obtains:

$$\tan\phi = 0 \tag{79}$$

and, as follows from Eqs. (77) and (78):

$$b_{22} = o \tag{80}$$

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Now it is obvious that along the asymptotic line tile curvature b_{11} can be selected arbitrarily without affecting the parameters of the internal geometry g_{ij}, and consequently, the stress. Indeed, since $b_{22} = 0$, b_{11} is eliminated from Eq. (71). In addition to that, as follows from Eq. (72):

$$\frac{\partial b_{11}}{\partial \psi^{(2)}} - \frac{\partial b_{12}}{\partial \psi^{(1)}} = \Gamma_{12}^1 b_{11} + (\Gamma_{11}^2 - \Gamma_{11}^1) b_{12} \tag{81}$$

$$\frac{\partial b_{12}}{\partial \psi^{(2)}} = \Gamma_{22}^1 b_{11} + (\Gamma_{22}^2 - \Gamma_{21}^1) b_{12} \tag{82}$$

The (derivative $\partial b_1, /\partial \psi_1$ is not defined, i.e. that asymptotic line ψ_1 coincides with the characteristic of the partial differential equations (81) and (82).

This means that the curvature b_{11} can be chosen arbitrarily along the asymptotic lines of the surface without effecting any parameters of the film including stresses. In other words, an unlimited growth of the curvature b_{11} may be consistent with the unboundedness of stresses and it can be associated with the formation of wrinkles along the asymptotic lines.

So far we were concerned with elastic continua. Turning to fluids, one should recall that their stresses depend only upon the velocities, but not upon the displacements. That is why an unlimited growth of any component of the displacement vector (M), or of the tensor-gradient (59) is consistent with the unboundedness of stresses, and it can be associated with the Lagrangian turbulence.

In terms of velocities, the situation is different. In order to demonstrate that, recall that in viscous fluid the stress tensor depends upon the velocity gradient ∇ v via the time derivative of the strain tensor (57). The velocity gradient ∇ v has the same type of structure as the vector-gradient ∇ r: it can be {Decomposed into a symmetric tensor of the rate. of strain:

$$\dot{\epsilon} = \frac{1}{2}(\nabla \mathbf{v} + [\nabla \mathbf{v}]^T) \tag{s3}$$

and an anti-symmetric tensor:

$$\omega = \frac{1}{2}[\nabla \mathbf{v} - (\nabla \mathbf{v})^T] \tag{s4}$$

which is equivalent to the vector of vortex:

$$\omega = \frac{1}{2} \operatorname{Curl} \mathbf{v} = \frac{1}{2} \nabla \times \mathbf{v} \tag{85}$$

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while

$$\nabla \mathbf{v} \;=\; \dot{\epsilon} \;+\; \omega \tag{86}$$

Since

$$\nabla \times \nabla \mathbf{v} \equiv 0 \text{ and } \nabla \cdot \nabla \times \nabla \mathbf{v} \equiv 0 \;\; (\text{even if}) \;\; \nabla \times \nabla \mathbf{v} \neq 0. \tag{87}$$

one comes to the same conclusion as in the case of the vector-gradient ∇ $\mathbf{r}$ (see Eqs. (63) and (65): nine components of the tensors $\dot{s}$ and ω are coupled by three compatibility equations. Hence, six components of the rate of strain tensor $\dot{\epsilon}$ uniquely define the velocity gradient, and for that reason, an unlimited growth of the vortices in viscous fluids would lead to unlimited growth of stresses.

The situation becomes different in inviscid fluids where stress is defined only by a scalar - the divergency $\nabla \times$ v. But since any velocity field can be uniquely defined based upon two independent components of its gradient ∇ v, which are the divergency ∇ x v and the vorticity $\nabla \times$ v, one concludes that an unlimited growth of vorticity in inviscid fluid may not lead to unbounded stresses. This conclusion can be loosely applied to motions of viscous fluids characterized by high Reynolds number when viscous stress are ignorable in comparison to the inertia forces. In this case an "unlimited" growth of vortices can be associated with turbulence.

Thus, in this section we have analyzed a possibility "in principle" of an unlimited decrease of scale of motions in continua from the viewpoint of a consistency of this type of instability with the boundedness of stresses and energy. This means that if such an instability exists, it can be found only in one-or two-dimensions] elastic models, or in fluid motions with high Reynolds number.

b. *Failure of Hyperbolicity in Distributed Systems*

Mathematical models of continua are based on the assumption that the functions describing their states can be differentiated "as many times as necessary" at any point exclusive of some special surfaces of discontinuities simulating shock waves or coinciding with tile characteristics of the governing equations. In other words, these functions must be at least piece-wise differentiable. From the physical viewpoint it means that any point as a center of mass of an infinitesimal volume represents all the properties of this volume. Obviously, the assumption about smoothness of the functions allows us to use tile

mathematical technique of differentiable equations.

However, this artificial mathematical limitation follows neither from the principles of mechanics nor from the definition of a continuum. The price paid for such a mathematical convenience is instability (in tilt' class of smooth functions) of the solutions to the corresponding governing equations in some regions of the parameters. T],is instability is characterized by unlimited decrease of the scale of the motions, in the course of which the derivatives of the corresponding functions tend to infinity although the functions them - selves remain finite. In other words, the solution tends to "go out" from the class of different iable functions.

Most of the instability phenomena leading to unlimited decreasing of the scale of continua motions are associated with the failure of hyperbolicity of the corresponding governing equations, i.e., with tilt' appearance of imaginary characteristic speeds. (Zak, 1982b,c)

In order to illustrate this, we will start with the governing equations of motion of elastic bodies in the following form:

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\left[\frac{\partial \Pi}{\partial \left(\frac{\partial u_i}{\partial x_j}\right)}\right]^+ F_i, \quad i = 1, 2, 3 \tag{88}$$

where u, are the displacements, 11 is the potential energy of strains, ρ is the density, F_i are the external forces, and x_i are the material coordinates, posing the initial value problem:

$$u_i^\circ = \left. U: \right|_{t=0} \begin{cases} = \frac{1}{\lambda^2} \sin \lambda_0 x_1 & \text{if } |x_1| \le x_0 \\ = 0^\circ & \text{if } |x_i| > x_0 , i = 1, 2, 3 \end{cases} \tag{s9}$$

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} i = 1, 2, 3. \tag{90}$$

with the parameter λ_0 can be made as large as necessary, i.e.,

$$\lambda_0 \to \infty \tag{91}$$

The region of the initial disturbance can be arbitrarily shrunk, i.e.,

$$|x_0| \to 0 \tag{92}$$

Consequently, the initial disturbances u_i and their first derivatives $\partial u_i / \partial x_i$ can be made as small as necessary. It means that for the corresponding infinitesimal period of time Δt_0 the equations (88) can be linearized and the solution subject to the initial conditions (89) can be sought in the form: $u_j = (x_i, t)$, i.e.,

$$\rho_0 \frac{\partial^2 u_i}{\partial t^2} = \sum_{j=1}^{3} a_{ij} \frac{\partial^2 u_j}{\partial x_1^2} , \quad \text{while} \quad \left. \frac{\partial u_j}{\partial x_j} \right|_{j \neq 1} \equiv 0 \tag{93}$$

where

$$a_{ij} = \left. \frac{\partial^2 \Pi}{\partial (\frac{\partial u_i}{\partial x_1}) \partial (\frac{\partial u_j}{\partial x_1})} \right|_{\frac{\partial u_i}{\partial x_i}, \frac{\partial u_j}{\partial x_1} = 0} \tag{94}$$

Let us assume that one of the eigenvalues of the matrix a_{ij} is negative:

$$\lambda_1 < 0 \tag{95}$$

Then the solution to the equation (93) will contain the term:

$$\frac{1}{\lambda_0^2} e^{\lambda_0 \sqrt{\frac{|\lambda_1|}{\rho_0}} \Delta t} \sin \lambda_0 x , \tag{96}$$

which tends to infinity if $\lambda_0 \to \infty$ within an arbitrary short period of time Δt_0 and within an infinitesimal volume around the point x_i. Hence, one arrives at the following situation:

$$|u_i| \to \infty, \tag{97}$$

in spite of the fact that

$$|u_i| \Big|_{t=0} \to 0 \tag{98}$$

However, strictly speaking, because of utilization of the governing equation (88) in a linearized form, the condition (98) must be weakened:

$$|u_i| \# O \quad \text{if} \quad |u_i| \Big|_{t=0} \to = 0 \tag{99}$$

The formula (99) shows that the appearance of negative eigen-values of the matrix (94), and consequently, imaginary characteristic roots of the governing equation (88) (failure of its hyperbolicity) leads to the violation of a continuous dependence between the initial an'd transient disturbances during an arbitrary short period of time and within an arbitrarily selected volume. This type of instability was first observed by .7, Hadamard in connection wit h the ill-posedness of the Cauchy problem for the Laplace equation. Further results with application to the instability of a string, film and free surfaces of elastic bodies were reported by Zak, (1982b,c.)

The result formulated above was obtained under specially selected initial conditions (S9), but it can be generalized to include any initial conditions. Indeed, for equations (93) let the initial conditions be arbitrarily defined by:

$$|u_i|_{t=0} = u_i^{00} \qquad (100)$$

and the corresponding solution is:

$$u_i = f_i(x, t) \qquad (101)$$

By altering the initial conditions to:

$$u(0, t) = u_i^0 + u_i^{00} \qquad (102)$$

where u_i is defined in (S9), we observe from the preceding argument by superposition that vanishingly small change in the initial conditions would lead to unboundedly large solutions.

To obtain a geometrical interpretation of the above described instability, let us turn to expression (97) of the solution and note that if the second derivatives $\partial^2 u_i/\partial t^2, \partial u_i/\partial x_i^2$ are of order λ_0, then the first derivatives $\partial u_i/\partial t, \partial u_i/\partial x_i$ are of order 1, and u_i are of order $1/\lambda_0$. Hence, the period of time Δt_0 can be selected in such a way that the second derivatives will be as large as necessary, but the first derivatives and u_i are still sufficiently small. Taking into account that the original governing equation (S8) is quasi-linear with respect to the second derivatives and, therefore, the linearization does not impose any restrictions on their values, one can conclude that the Linearized equation (93) is valid for the solution during the above- mentioned period of time At., Turning to the formula (97) one can now interpret the solution by the function having an infinitesimal amplitude and changing its signs with an infinite frequency ($\nu = \lambda_0 \to \infty$). The first derivatives of this function $\partial_i/\partial t, \partial_i/x_i$ can be small and change their signs by finite jumps (with the same infinite frequency ν) so that the second derivatives $\partial^2 u_i/\partial t^2, \partial^2 u_i/\partial x_i^2$ at the points of such jumps are infinite. Thus, within an arbitrary small volume there is located an arbitrary large number of points at which tile strains have jumps. From the mathematical point of view, the function describing such a field of displacements u_i is considered as a continuous but non-differentiable function. This function can be simulated, for instance by the function with a multivalued derivative.

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C. THE CRITERIA OF HADAMARD'S INSTABILIT%

Let us fix an arbitrary point M and an arbitrary direction x_1 at this point in an elastic body. According to the above-formulated result, the instability at the point M in the x-direction results from the negative eigenvalues of the matrix:

$$a_{ij} = \left\{ \frac{\partial^2 \Pi}{\partial\left(\frac{\partial u_i}{\partial x_1}\right)\partial\left(\frac{\partial u_j}{\partial x_1}\right)} \right\}\Bigg|_{\frac{\partial u_i}{\partial x_1}, \frac{\partial u_j}{\partial x_1}} \tag{103}$$

Assuming that unperturbed state at this point is characterized by the initial stresses:

$$T^0_{1j} \neq 0, \ T^0_{2j} = 0, \ T^0_{3j} = 0, \ j = 1, 2, 3 \tag{104}$$

but zero strains:

$$\left(\frac{\partial u_i}{\partial x_1}\right)_0 = 0, \ i.e., \epsilon^0_{11} = 0, \ \gamma^0_{12} = 0, \ \gamma^0_{13} \tag{105}$$

let us utilize the following expression for a variation of the specific potential energy from the initially stressed defined by (104).

$$\delta\Pi = T_{11}\delta\epsilon_{11} + T_{12}\delta\gamma_{12} + T_{13}\delta\gamma_{13} = \cdots etc. = T_{ij}\,\delta\epsilon_{ij} \tag{106}$$

Taking into account that:

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2 + \left(\frac{\partial u_3}{\partial x_1}\right)^2\right],$$

$$\gamma_{1i} = \frac{\partial u_i}{\partial x_1} + \cdots etc. \quad (i = 1, 2, 3), i.e.,$$

$$\delta\epsilon_{11} = \left(1 + \frac{\partial u_1}{\partial x_1}\right)\delta\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_1}\delta\frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_1}\delta\frac{\partial u_3}{\partial x_1},$$

$$\delta\gamma_{1i} = \delta\frac{\partial u_i}{\partial x_1} \quad (i = 1, 2, 3)$$

$$T_{1i}\Bigg|_{\frac{\partial u_i}{\partial x_1} = 0} \tag{107}$$

one obtains for

$$\frac{\partial u_i}{\partial x_1} \to 0 :$$

$$a_{11} = T^0_{11} + \frac{\partial T_{11}}{\partial\epsilon_{11}}, \tag{108}$$

$$a_{22} = T^0_{11} + \frac{\partial T_{12}}{\partial\gamma_{12}} = T^0_{11} + \frac{1}{2}\frac{\partial T_{12}}{\partial\epsilon_{12}}, \tag{109}$$

$$a_{33} = T^0_{11} + \frac{\partial T_{13}}{\partial\gamma_{13}} = T^0_{11} + \frac{1}{2}\frac{\partial T_{13}}{\partial\epsilon_{13}}, \tag{110}$$

$$a_{12} = a_{21} = a_{13} = a_{31} = a_{23} = a_{32} = 0 \tag{111}$$

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where the stresses T_i are related to the lot-al cartesian coordinates x_1, x_2, x_3 at the point M.

Now the eigenvalues of the matrix (106) can be written in the form:

$$\lambda_i = a_{ii} \text{, i.e.,}$$

$$\lambda_1 = T_{11}^0 + \frac{\partial T_{11}}{\partial \epsilon_{11}} \tag{112}$$

$$\lambda_2 = T_{11}^0 + \frac{\partial T_{12}}{\partial \gamma_{12}} = T_{11}^0 + \frac{1}{2}\frac{\partial T_{12}}{\partial \epsilon_{12}} \tag{113}$$

$$\lambda_3 = T_{11}^0 + \frac{\partial T_{13}}{\partial \gamma_{13}} = T_{11}^0 + \frac{1}{2}\frac{\partial T_{13}}{\partial \epsilon_{13}}. \tag{114}$$

Hence, the criteria of instability are:

$$T_{11}^0 < -\frac{\partial T_{11}}{\partial \epsilon_{11}} \text{ ,} \tag{115}$$

$$T_{11}^0 < -\frac{\partial T_{12}}{\partial \gamma_{12}} \text{ ,} \tag{116}$$

$$T_{11}^0 < -\frac{\partial T_{13}}{\partial \gamma_{13}} \text{ .} \tag{117}$$

Each inequality leads to the failure of differentiability of the corresponding component of strains: $\epsilon_{11}, \epsilon_{12}, or \epsilon_{13}$, while the potential energy $\Pi(\epsilon_{1i})$ has a local maximum.

Recall that all the shove-fornmlated results are related to an arbitrary point M_0 and -arbitrary selected direction xl, with the unidirectional initial stress T_{11}^0.

In the general case when all the components of the initial stresses are non-zero:

$$T_{ij}^0 \neq 0 \tag{11s}$$

one can decompose them into spherical and deviatoric parts:

$$T_{ij}^0 = \frac{1}{3}T_0 E + T_{ij}^{0*}, \quad T_{ij}^{0*} = \text{Dev}T_{ij}^0 \tag{119}$$

where E is the unit tensor, and

$$To = \frac{1}{3}\sum_{i=1}^{3} T_{ij} \text{ ,} \quad \sum_{i=1}^{3} T_{ij}^* = O \tag{120}$$

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Now Eq.(106) can be rewritten in the following form:

$$\delta\Pi = T_0\delta\epsilon_0 + T_{11}^*\delta\epsilon_{11} + T_{12}^*\delta\epsilon_{12} + \cdots etc. = T_0\delta\epsilon_0 + T_{ij}^*\delta\epsilon_{ij} \qquad (121)$$

where ϵ_0 is the spherical part of tile strain tensor:

$$\epsilon_0 = \frac{1}{3}\sum_{i=1}^{3}\epsilon_{ij} \qquad (122)$$

and instead of Eqs. (10S) - (110), one obtains:

$$a_{11} = T_{11}^0 + \frac{\partial T_{11}^*}{\partial\epsilon_{11}}$$

$$a_{22} = T_{11}^0 + \frac{1}{2}\frac{\partial T_{12}^*}{\partial\epsilon_{12}}$$

$$a_{33} = T_{11}^0 + \frac{1}{2}\frac{\partial T_{13}^*}{\partial\epsilon_{13}} \qquad (123)$$

Consequently, the sufficient conditions of the instability in some directions at the fixed point for an isotropical elastic material for which the derivatives $\partial T_{ij}/\partial\epsilon_{ij}$ do not depend on a selected direction x_1 :

$$\hat{T}_{11}^0 < -\frac{\partial T_{11}}{\partial\epsilon_{11}}, \qquad (124)$$

$$\hat{T}_{11}^0 < -\frac{1}{2}\frac{\partial T_{12}}{\partial\epsilon_{12}}, \qquad (125)$$

$$\hat{T}_{11}^0 < -\frac{1}{2}\frac{\partial T_{13}}{\partial\epsilon_{13}}, \qquad (126)$$

where $\hat{T}_{11}^0$ is one of the principle deviatoric stresses,

The instability emerges in any direction if these inequalities are valid for all the principle deviatoric stresses $\hat{T}_{ii}^0$:

$$\hat{T}_{ii}^0 < -\frac{1}{2}\frac{\partial T_{ij}}{\partial\epsilon_{ij}} \quad (i + j), \qquad (127)$$

because usually

$$\frac{\partial T_{ii}}{\partial\epsilon_{ii}} > \frac{\partial T_{ij}}{\partial\epsilon_{ij}} \quad (i \neq j). \qquad (128)$$

For the Hook's material the criteria of the instability are expressed in terms of Young's modules E and Poisson's ratio ν since

$$\frac{\partial T_{ii}}{\partial\epsilon_{ii}} = \frac{E(I - v)}{(\nu + 1)(1 - 2\nu)} \quad \frac{\partial T_{ij}}{\partial\epsilon_{ij}} = \frac{E}{\nu + 1} = 2G , \qquad (129)$$

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if the initial stress tensor is spherical ($T_{ij}^0 = p$), where E, G, and μ are the Young's and shear modulic, and v is the Poisson's ration, respectively.

d.   Boundaries of Applicability of the Classical **Models** of **Distributed** Systems

All the results discusser above were based on formal analysis of mathematical models of elastic materials, and their practical usefulness has to be demonstrated. The most obvious and visual application of these results can be fund in the area of one and two-dimensional models such as strings, membranes, etc. whose states are defined not only by internal geometry (strains), but also by external geometry (shape). As shown in the section a), in this model, unlimited decrease of the scale of motions may be consistent with the boundedness of stresses and energy. The problem of the shape instability there occurs as a result of any local compression and manifests itself in wrinkling in the course of which the shape looses its smoothness.

Examples

(a) For one-dimensional continua such as an ideally flexible inextensible string, two types of the characteristic speeds are obtained (Zak, M., 1968):

$$\lambda_{1,2} = \pm \left(\frac{T}{\rho} \right)^{\frac{1}{2}} , \qquad (130)$$

$$\lambda_{3,4} = \pm \left(\frac{T}{\rho} + \frac{F_n}{\Omega} \right)^{\frac{1}{2}} , \qquad (131)$$

where T is the tension, p **is the** linear density, Ω is the first curvature of the string's shape, and F_n is the normal component of the external tracking force. These characteristic speeds correspond to discontinuities of the curvature and twist of the string, respectively (Fig. 4).

These conditions of the instability of the string's shape following from the failure of hyperbolicity are given in the form:

$$T < O , \qquad (132)$$

$$T < \frac{F_n}{\Omega} \qquad (133)$$

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The inequality (131) expresses the well-known fact that a compressed string is unstable (the 10ss of the stability of the first curvature, Fig 1). The shape of such a string cannot be described by differentiable functions, and, theoretically, that string can be rolled up in a point. The inequality (132) shows that even a stretched string can be unstable, if subjected to the corresponding tracking force (the loss of tile stability of the twist).

These results are generalized to a one-dimensional, ideally flexible pipe within which an ideal fluid flows (Zak, M.1982b, c)

$$\lambda_{1.2}^{1} = \frac{\rho^1}{\rho + \rho^1} u \pm [\frac{T}{\rho + \rho^1} - \frac{\rho\rho^1}{(\rho + \rho^1)^2} u^2]^{1/2} ,$$

$$\lambda_{3.4}^{1} = \frac{\rho^1}{\rho + \rho^1} u \pm \left(\frac{T}{\rho + \rho^1} + \frac{F_n}{(\rho + \rho^1)\Omega} - \frac{\rho\rho^1}{(\rho + \rho^1)^2} u^2 \right)^{1/2} , \tag{134}$$

where T is the tension referred to the entire pipe's cross-section, ρ^1 is the linear density of the fluid, and u is the velocity of the fluid.

Then, the condit ions of the failure of hyperbolicity are given by

$$T < \frac{\rho\rho^1}{(\rho + \rho^1)} u^2 , \tag{135}$$

$$T < \frac{F_n}{\Omega} + \frac{\rho\rho^1}{(\rho + \rho^1)^2} u^2 , \tag{136}$$

This means that a flow within the pipe destabilizes its shape. In order to illustrate the last results, let us consider a vertical, ideally flexible, inextensible pipe with a free lower end suspended in the gravity field, Assuming that the flow within this pipe has constant velocity, U., let us define the area of the instability (Fig. 5). The tension T referred to t he entire pipe's cross-section is given by

$$T = \rho g \zeta (l - x) \tag{137}$$

where 1 is the length of the pipe, x is the coordinate along the length of the pipe, and ζ is the ratio of the area of the cross-section occupied by the pipe's walls relative to the entire cross-sectional area, Substituting (3.9) into (3.7), one obtains the unstable area of the pipe:

$$l \geq x \geq l - \frac{u_0^2}{g\zeta(1 + \epsilon)} , \quad \epsilon = \frac{\rho}{\rho^1} . \tag{138}$$

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Hence, for the ideal flexible pipe, the free end is always unstable. (Such a phenomenon is well known from the experiments.) In the limit case $u_0 \to O$, when tile pipe can be considered as a string, the unstable area is concentrated around the free end. As shown by Zak, M. 1970, 1983, such an instability manifests itself in an accumulation of energy at the free end (snap of a whip).

(l)) For two-dimensional continua, such as membranes, films, and nets, as shown by Zak. M., 1979, the characteristic speed corresponding to discontinuities of the shape (i.e.. the coefficients of the second fundamental form) is given by

$$\lambda_{1,2} = \pm \left(\frac{T_n}{\rho} \right)^{1/2} \tag{139}$$

where T_n is the tension normal to the front of the wave of a discontinuity.

Hence, the failure of hyperbolicity emerges in the region where at least one of the principal stresses is negative. Such a failure manifests itself in formation of wrinkles. The wrinkles can be observed, for instance, in the course of shearing, twisting, or bending of a membrane (Fig. 6). If both of the principal stresses are negative, then even the lines of wrinkles lose their smoothness, ant] a membrane can be rolled up in a point.

Recall that in contrast to one dimensional continua where the shape parameters (curvature and twist) can be changed independently from the elongations, in two dimensional continua there are some limitations imposed on the changes of the shape in the form of the equations of compatibility with the changes of strains (the Gauss equations (71) and (72). As follows from Eq. (78) - (82), at the points of negative Gaussion curvature, there are two directions "of possible shape wave propagations (Fig. 78). At the point of zero Gaussion curvature, there is only one such a direciton (Fig, 7c). At last, at the points of positive Gaussion curvature, the shape discontinuities are impossible, (Figure 7d).

Thus the instability of the shape defined in terms of the coefficients of the second fundamental form b_{ij} is possible only if a compression occurs in the direction normal to the asymptotic line of the surface at the corresponding point.

Slightly different, criteria of the Hadamard's instability can be obtained for liquid films considered as two-dimensional continua, (Zak, M., 1985a).

The Hadamard's instability for three-dimension continua were studied by Zak, M., 1982a,b,f. In this paper we will focus our attention on the instability of a surface separating an elastic body and ideal fluid. (Fig. S).

As shown by Zak, M., 1982b, the characteristic speed of waves transporting discontinuities of the surface shape is expressed as:

$$\mu = \frac{\rho_2}{\rho_1 + \rho_2} v \pm \sqrt{\frac{1}{\rho_1 + \rho_2} T_{nn} + \frac{E}{2(1 + \nu)} - \frac{\rho_1 \rho_2}{(\rho_1 + \rho_2)^2} v^2} \qquad (140)$$

in which ρ_1, E, ν and$_{nn}$ characterize the density, Young's modules, the Poisson's ratio and the stress normal to the front of the propagating wave of the elastic body, and ρ_2 and v characterize the density and the velocity of the fluid.

Hence, the Hadamand's instability occurs if:

$$T_{nn} < \frac{\rho_1 \rho_2}{(\rho_1 + \rho_2)^2} v^2 - \frac{E}{2(1 + \nu)} \qquad (141)$$

As a particular case of Eq. (140), one can arrive at the Hadamard's instability of surface of tangential jump at velocity in an inviscid fluid (Fig, 9)

$$\lambda = \frac{1}{2} \left[(u_2 - u_1) \pm \sqrt{-(u_2 - u_1)^2} \right] = = \frac{1}{2}(u_2 - u_1)(1 \pm i) \qquad (142)$$

This is a well-known result stating that tangential jumps of velocities in inviscid fluids are always unstable. (In fluid mechanics this phenomenon is called Kelvin-Helmholtz instability).

4. **Cumulative Effects**

a. **Degenerating Hyperbolic Equations**

A cumulative effect can be introduced as a pre-instability state which is associated with the change of type of governing equations from hyperbolic to parabolic when at least one of the characteristic speed becomes zero. Actually this stock represents the boundary for the Hadamard's instability, and depending on how the motion approaches this boundary, it may remain stable, or unstable. The simplest example of this type of situation is the governing equation for a vertical ideally flexible inextensible string with a free lower end suspended in a gravity field, Fig, 8. Projecting this equation into the horizontal direction. one arrives at the governing equations for small transverse motion of the string:

$$\frac{\partial^2 x}{\partial t^2} + \frac{T}{\rho} \frac{\partial^2 x}{\partial \psi^2} = 0 \qquad (143)$$

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with the characteristic speeds:

$$\mu = \pm\sqrt{\frac{T}{\rho}} \tag{144}$$

Since the tension of the string T vanish at the free end

$$T = 0 \quad \text{at} \quad S = l \tag{145}$$

where l is the leng h of the string, the characteristic speeds 144) vanish too at $S = 1$, and therefore, Eq. (143) degenerate into parabolic type at the very end of the string.

As a second example, consider a one-dimensional model of the shear wave propagation in a soil column of the height H:

$$\rho\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(G\frac{\partial u}{\partial x}\right) \tag{146}$$

where ρ is tile density, u is the horizontal (Displacement, G is the shear modules, t is time, and x is the vertical coordinate with the origin at the surface.

Ignoring the small shear stresses at the surface, the shear modules can be taken in the following form:

$$G = 0.5\rho g x . \tag{147}$$

Since

$$G = 0 \quad \text{at} x = 0 , \tag{149}$$

Eq. (146) degenerates into parabolic type at the soil surface.

For the sake of concreteness, we will investigate the solution to Eq. (146) subject to the initial and boundary conditions formulated below:

$$u(x, 0) = \varphi(x) \tag{150}$$

Thus, it is assumed that the soil column is fixed at $x = H$ and there is no shear stress at the surface, i.e., at $x = 0$.

One should notice that for simplicity in this model all the damping and creep effects are ignored.

b. Uniqueness of the Solution

Let US assume that there exists two solutions of the problem under consideration: $u'(x,t)$ and $U''(x, t)$ and let us examine the difference:

$$u^*(x, t) = u'(x, t) - u''(x, t) \tag{151}$$

The function u^* (x,t) satisfied Eq. (146) with additional homogeneous conditions:

$$\rho \frac{\partial^2 u^*}{\partial t^2} = \frac{\partial}{\partial x}\left(G\frac{\partial u^*}{\partial x}\right) \tag{152}$$

$$u^*(x, 0) = , \frac{\partial u^*}{\partial t}(x, 0) = 0 \tag{153}$$

$$u^*(H, t) = o, \frac{\partial u^*}{\partial x}(0, t) = 0 \tag{154}$$

For the total energy, one gets:

$$E(t) = E(0) = \frac{1}{2}\int_0^{H}\left\{G\left(\frac{\partial u^*}{\partial x}\right)^2 + \rho\left(\frac{\partial u^*}{\partial t}\right)^2\right\}\bigg|_{t=0} dx = 0 \tag{155}$$

If the solution is sought in the open interval

$$0 < x \leq H \tag{156}$$

which does not include the surface point $x = O$, then the uniqueness of the solution is obvious.

However, this proof cannot be applied to the closed interval

$$0 \leq x \leq H \tag{157}$$

which includes the surface point $x = O$., Indeed, in this case according to equation (148):

$$G = 0 \text{ at } x = 0 \tag{158}$$

and any arbitrarily selected derivative $\partial u_*/\partial x$ at $x = O$ will satisfy the equality in Eq. (155).

Thus, for the closed interval (157), the uniqueness of the solution can be guaranteed only in the claw of functions having continuous [derivative $\partial u^*/\partial x$, otherwise the infinite number of different solutions can be offered to satisfy equations (146) with the conditions (145) and (150) in the interval (157). As will be shown in the following, the artificial mathematical restriction about the continuity of the derivative ∂u ● /dx excludes such import ant physical phenomena as cumulative effect.

From the mathematical point of view, the singularity at the point $x = 0$ is associated with the fact that the original equation is hyperbolic in the open interval (136), but degenerates into a parabolic equation at the point $x = 0$. The physical meaning of this singularity will be discussed in the following section.

c.   Stability of the Solution

Starting with the conditions (149), let us assume that

$$\varphi(x) \begin{cases} > 0 & \text{for } 0 < x_1^*, < x < x_2^* < H \\ = o & \text{for } x < x_1, \text{ and } x > x_2 \end{cases} \tag{159}$$

$$\psi(x)|_{0 \leq x \leq H} = o \tag{160}$$

$$\alpha(t) = 0, t \geq 0 \tag{161}$$

i.e., we consider an initial disturbance in a local interval $[x_1^*, x_2^*]$ contained within the interval $[0, H]$.

From the differential equation of the characteristics, one finds the equations of the characteristics passing through x_1 and x_2:

$$t' = \int_{x_1}^{x_2^*} \frac{d\epsilon}{\sqrt{G(\epsilon)/\varphi}}, \quad t'' = \int_{x_2}^{x_2^*} \frac{d\epsilon}{\sqrt{G(\epsilon)/\varphi}} \tag{162}$$

$$0 < x_1 < x_2 < H$$

Here x_1 and X_2 are the coordinates of the leading and trailing fronts of the discontinuity wave of derivatives $\partial^2 u/\partial t^2$ and $\partial^2 u/\partial x^2$, where

$$x_1 = x_1^*, \quad x_2 = x_2^* \text{ at } t = 0$$

A singular solution coincident for both characteristics holds for

$$x_1 = x_2 = 0$$

because

$$\frac{dx_1}{dt}\bigg|_{x_1=0} = \frac{dx_2}{dt}\bigg|_{x_2=0} = 0 \tag{163}$$

Two cases may arise: (A) The improper integral

$$\int_{x}^{x_0} \frac{d\eta\,d\epsilon}{\sqrt{G(\eta)/\rho}} \;<\; \infty \text{ for } x \to 0 \tag{164}$$

i.e., converges, which then means that coincidence of tilt' characteristics occurs for finite $t = t^*$. Then

$$\left|\frac{\partial u}{\partial t}\right| \to \infty \text{ for } t \to t^* < \infty \tag{165}$$

From the mathematical viewpoint this instability predicts a cumulation of the shear strain energy at the soil surface $x = 0$. At tile same time, it illustrates the ambiguity in the solution which has been remarked in the investigation of equation (15).

(El) If the improper integral (164) diverges then the characteristics (162) coincide at $t^* \to \infty$ and the cumulation effect does not occur.

For the particular case of soil where the shear modulus is given in the form (167) the integral (164) converges and the time $t_\bullet$ defining the moment of the formation of the shear strain energy cumulation at the soil surface is

$$t^* \;=\; 2\sqrt{\frac{2x_2^*}{g}}$$

In the general case when the shear modulus is a more complicated function of the elevation, the cumulative effect occurs if

$$G \;>\; \epsilon_1 x^{2+\epsilon_2}$$

where ϵ_1, ϵ_2 are arbitrarily small positive constraints, because then the integral (164) converges.

do **Snap of a Whip**

The results presented shove can be applied to Eq. (163) [Inscribing transverse oscillations of a vertical ideally flexible inextensible string with a free lower end suspended in a gravity field.

The tension of the string due to gravity is given by the following equation

$$T \;=\; \gamma(l - x) \tag{166}$$

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where γ, l are the specific weight and length of the st ring.

Referring to the formula (144), one concludes that the characteristic speed of transverse displacements tends to zero at the free end:

$$T|_{x=l} = 0 , \tilde{\gamma}_{3,4} \to 0 \text{ if } x \to l . \tag{167}$$

In other words, for the small transverse displacements of the string, the governing equation is of hyperbolic type only in the open interval, excluding the end:

$$0 \leq x < l \tag{168}$$

As shown in the previous section, in this open interval there exists a unique stable solution. However, in the closed interval, including the end

$$0 \leq x \leq l , \tag{169}$$

the solution is not unique and tilt're are unstable solutions if the improper integral

$$\int_{0}^{x} \frac{d\xi}{[T(\xi)/\rho]^{1/2}} , \tag{170}$$

converges for $x \to 1$.

This result has a very clear physical interpretation: Suppose that an isolated transverse wave of small amplitude was generated at the point of suspension (Figure 5b). The speed of propagation of the leading front of the transverse wave will be smaller than the speed of the trailing front because the tension decreases from the point of suspension to the free end (see equations (144) and (166). Hence, the length of the above wave will be decreasing and in some cases (see (170)) will tend to zero. Then according to the law of conservation of energy, the specific kinetic energy per unit of length will tend to infinity producing a snap (snap of a whip).

It can be easily verified by substituting (166) in (170) that for the string in the gravity field the integral (170) converges, i.e., the instability in the form of snap occurs.

The same type of instability as a result of cumulation of energy near the boundary of the failure of hyperbolicity can exist in two. and three-dimensional models in the domains where the inequalities (116), (11 7), (118) are close enough to the corresponding equalities. Zak, M. (1982 c).

e.   Failure of Lipschitz Conditions

The cumulative effects are accompanied by a very interesting mathematical phenomenon: failure of Lipschitz conditions for the differential equations of characteristics:

$$\frac{ds}{dt} = \dot{s} = \lambda(s) \tag{171}$$

Indeed, if the characteristic speed follows from Eq. (143) or Eq. (146), i.e.

$$\lambda = \pm\sqrt{\frac{T}{S}} \text{ , or } \lambda = \pm\sqrt{\frac{G}{S}} \text{ , respectively} \tag{172}$$

then

$$\left|\frac{\partial\lambda}{\partial s}\right| = \left|\frac{-1}{2\lambda}\frac{\partial\lambda^2}{\partial s}\right| \to 0 \text{ at } S \to S_0 \tag{173}$$

if

$$\left|\frac{\partial\lambda^2}{\partial S}\right| > 0 \text{ at } S \to S_0$$

As follows from Eqs. (147), and (166),

$$\left|\frac{\partial\lambda^2}{\partial s}\right| = 0.5g > 0 \text{ , and } \left|\frac{\partial\lambda^2}{\partial s}\right| = g > 0 \text{ , respectively} \tag{174}$$

Hence, the loss of tile uniqueness of the solution to Eqs. (143) and (146) can be formally associated with the failure of the Lipschitz condition at the point where the characteristics coincide.

In general, failure of the Lipschitz conditions in dynamics was analyzed by Zak, M. (19ss, 1992, 1993a,b).

5.   **Comments** On Other Types of Instability In Dynamics

As follows from the previous section, the Hadamard Instability occurs in idealized models such as elastic bodies, or ideal fluids where the energy dissipation can be ignored. Tile main property of this type of instability is that the solution becomes unbounded during a finite time interval ($t < \infty$). However, tilt'lt' are many other types of instability (especially, in fluid dynamics) which also leads to unlimited decrease of the scale of motion, although they are not so "strong" as the Hadamard instability: the solution becomes unbounded only at $t \to \infty$. Since all of this Lyapunov-type of instabilities are well represented in literature, we will give here only a brief description of them.

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a.   Thermal Instability arises when a fluid is heated from below. When the temperature difference across the fluid layer is great enough, the stability effects of viscosity and tilt'mlal conductivity are overcome by the destabilizing buoyancy, and the instability occurs in the form of a thermal convection.

b.   Centrificial Instability occurs in a fluid owing to the dynamical effects of rotation or of streamline curvature. For instance, as shown by Rayleigh, an invicid flow between two rotating coaxial cylinders is unstable if the angular momentum $|r^2\Omega|$ should decrease anywhere inside the internal $r_1 < r < r_2$, where Ω is the angular velocity of rotation of the fluid, r_1, and r_2 are the radii of tile coaxial cylinders.

It can be demonstrated that in general, centrificial instability arose from adverse distributions of angular momentum.

Rayleigh-Taylor instability derives from the character of the equilibrium of an incompressible heavy fluid of variable density. For instance, it is shown, that in the case of variable density of exponential distribution

$$\rho = \rho_0 \; e^{\beta z} \, , \; \beta \; = \; \text{Const} \tag{175}$$

where z is the vtv-tics] coordinate, the equil brium is unstable if

$$\beta \; > \; 0 \tag{176}$$

i.e., if the heavier layers are above the lighter layers,

c.   Reynolds Instability results from a disbalance between the inertial and viscus forces. It occurs when the Reynolds number (R) exceeds certain critical values which depend upon the type of a flow and its boundary conditions. For a particular case of inviseid shear flow $(R \rightarrow \infty)$ with parallel streamlines, Rayleigh has shown that a necessary condition for instability is that the basic velocity profile should have an inflection print.

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III.   STABILIZATION PRINCIPLE

1.   **Instability** As Inconsistency Between Models and Reality

a.   General Remarks

It has been demonstrated in the previous section that there are some domains of dynamical parameters where the motion cannot be predicted because of instability of tile solutions to the corresponding governing equations. How can it be interpreted? Does it mean that the Newton's laws are not adequate'? or is there something wrong with our mathematical models? In order to answer these questions, we will discuss some general aspects of the concept of instability, and in particular, a degree to which it is an invariant of motion. We will demonstrate that instability is an attribute of a mathematical model rather than physical phenomenon, that it depends upon tile frame of reference, upon the class of functions in which the motion is described, and upon the way in whit-h the distances between the basic and perturbed solutions is defined.

b.   Instability Dependence Upon Metrics of Configuration Space

Let us turn to orbital instability discussed in the Section 2.2. The metric of configuration space where the finite-degree-of-freedom dynamical system with N generalized coordinates $q^i(i = 1, 20. . N)$ is represented by a unit-mass particle, was defined by Eqs. (27) and (28). Now there are at least two possible ways to define the distance between the basic anti disturbed trajectories. Following Synge, (1926), we will consider tile distance in kinematical and in kinematico-statistical sense. In the first case the corresponding points on the trajectories are those for which time t has the same value. In the second case tile correspondence between points on the basic trajectory C and a disturbed trajectory C'* is established by the condition that P (a point on C) should be the foot of the geodesic perpendicular let fail from P^* (a point on C*) on C, i.e., here every point of the disturbed curve is adjacent to the undisturbed curve (regardless of the position of the moving particle at the instant t). As shown by Synge, both definition of stability are invariant with respect to coordinate transformation, and in both cases the stability implies that the corresponding distance between the curves C and C* remains permanently small.

It is obvious that stability n the kinematical sense implies stability in the kinematico-

statical seine, but tile converse is not true. Indeed, consider the motion of a particle of unit mass on a plane under the influence of a force system derivable from a potent ial:

$$\Pi = -x + \frac{1}{2}y^2.$$ (177)

Writing down the equations of motion and solving them, we get:

$$x = \frac{1}{2}t^2 + At + B$$ (178)

$$y = c \, \text{Sin} \, (t + \alpha)$$ (179)

where A, B, C and D are constants of integration.

Let the undisturbed motion be:

$$x = \frac{1}{2}t^2 + t$$ 1s0)

$$y = 0$$ 1s1)

The motion is clearly unstable in the kinematical sense. However, from the viewpoint of stability in the kinematic o-statical sense, the distance between corresponding points is:

$$P \, P^* = y = C \, \text{Sin} \, (t + D)$$ (182)

remains permanently small if C is small. Hence, there is stability in the kinematico- statical sense.

Thus, the same motion can be stable in one sense, and unstable in another, depending upon the. way in which the distance between the trajectories is defined.

It should be noticed that in both cases, the metric of configuration space was the same (see Eqs. (27) and (28). However, as shown by Synge, 1926, for conservative systems. one can introduce a configuration space with another metric.

$$g_{mn} = (E - \Pi)\alpha_{mn}$$ (183)

where α_{mn} are expressed by Eq. (27), and E is the total energy.

The system of motion trajectories here consists of all the geodesics of the manifold. The correspondence between points on the trajectories is fixed by the condition that the arc $O^* \, P^*$ should be equal to the arc, OP, where O and O^* are arbitrarily selected origins on the basic trajectory and any disturbed one, respectively,

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As shown by Synge, the problem of stability here (which is called stability in the action sense) is that of the convergence of geodesics in Riemannian space. If two geodesics pass through adjacent points in nearly parallel directions, the distance between points cm the geodesics equidistant from the respective initial points is either permanently small or not. If not, there is instability. It appears that stability in the action sense may not be equivalent to stability in the kinematico-statical sense for distances which change the total energy E.

Turning to the example, Eq. (177), let us take the initial point O at the origin of coordinates and the initial point O^* on the y axis. Then the disturbance being infinitesimal, the (action) distance between corresponding points is:

$$P^* = (E - \sqcap)^{1/2} \ y \ = 2^{-1/2}(t \ + \ 1) \ C \ \mathrm{Sin} \ (t \ + \ D) \tag{184}$$

Hence, the motion is unstable in the action sense.

c.   Instability Dependence Upon the Frame of Reference

Dynamical instability depends not only upon the metric in which the distances between trajectories are defined, but also upon the frame of reference in which the motion is described. Such a dependence was already noticed above (see Eqs. (55)). In this section we will introduce and discuss an example which illustrates the dependence of criteria of hydrodynamics stability and the onset of turbulence upon the frame of reference.

The linear theory of hydrodynamic stability is based upon Eulerian representation of fluid motions in which the frame of reference is chosen a *priori*. Strictly speaking, such a representation provides criteria of stability for the velocity field rather than the fluid motion. The difference between these two types of stability was illustrated by Arnold (1988), who introduced flows with stable velocity fields and unstable trajectories (Lagrangian turbulence). If the classical (Eulerian) turbulence is associated with the instability of streamlines then it is reasonable to study this instability in a streamline frame of reference in which streamlines form a family of initially unknown Eulerian coordinates, while the remaining two Lagrangian coordinates are found from the compatibility conditions. Such a frame of reference is completely defined by the motion, and therefore, it contains a minimum of arbitrarily chosen parameters.

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First of all, we will show that criteria of stability in this frame of reference do not necessarily coincide with the classical criteria which are derived from t he Orr-Sommerfeld equation. For this purpose, we will introduce a small disturbance velocity field for incompressible plane flow in cartesian coordinates x. y:

$$V_x = \phi'(y)e^{i(\alpha x - \beta t)} , \quad V_y = -i\alpha\phi \; y)e^{i(\alpha x - \beta t)} , \quad \alpha, \beta = \text{const.} \tag{185}$$

where the prime denotes differentiation.

Tile angle θ between streamlines and tile x-direction is

$$\theta = \frac{i\alpha\phi}{V} e^{i(\alpha x - \beta t)} , \tag{186}$$

in which V(y) is the velocity profile of the basic flow. The orthogonal streamline coordinates ξ, ζ are found from the system:

$$\frac{\partial x}{\partial \xi} = H_1 \cos\theta , \quad \frac{\partial x}{\partial \zeta} = -H_2 \sin\theta , \quad \frac{\partial y}{\partial \xi} = H_1 \sin\theta , \quad \frac{\partial y}{\partial \zeta} = H_2 \cos\theta , \tag{187}$$

where H_1 and H_2 are the Lame coefficients defined by the compatibility conditions

$$(\partial^2 x / \partial \xi \partial \zeta = \partial^2 x / \partial \zeta \partial \xi \text{ etc.})$$

$$\left(\frac{\partial \theta}{\partial x} + \frac{\partial \theta}{\partial y} \tan\theta \right) H_1 = \tan\theta \frac{\partial H_1}{\partial x} - \frac{\partial H_1}{\partial y} \tag{1ss}$$

and

$$\left(-\frac{\partial \theta}{\partial y} + \frac{\partial \theta}{\partial x} \tan\theta \right) H_2 = \tan\theta \frac{\partial H_2}{\partial y} - \frac{\partial H_2}{\partial x} . \tag{189}$$

.4s follows from Eqs. (186 - 189), the coordinate transformation

$$x = x(\xi, \zeta, t) , \quad y = y(\xi, \zeta, t) \tag{190}$$

in general will depend on time. Hence, for the stream function one obtains

$$\psi = \phi(y) e^{i(\alpha x - \beta t)} = \phi[y(\xi, \zeta, t]e^{i[\alpha x(\xi,\zeta,t)-\alpha t]} , \tag{191}$$

i.e.

$$\left.\frac{\partial \psi}{\partial t}\right|^{x,y=\text{const}} \# \left.\frac{\partial \psi}{\partial t}\right|_{\xi,\zeta=\text{const}} \tag{192}$$

In other words, the stability criteria in frames x, y and ξ, ζ are not necessarily the same.

This preliminary conclusion provides motivation to analyze criteria of hydrodynamic stability in streamline coordinates.

Confining our investigation to a plane incompressible inviscid flow one (It'rives the momentum equations in streamline coordinates from the Lagrange equation:

$$\frac{d}{dt}\frac{\partial w}{\partial \xi} - \frac{\partial w}{\partial \xi} = \frac{1}{\rho}\frac{\partial p}{\partial \xi} \; ; \; \frac{\partial w}{\partial \zeta} = -\frac{1}{\rho}\frac{\partial p}{\partial \zeta} \; . \tag{193}$$

in which the kinetic energy

$$W = \frac{1}{2}H_1^2\xi^2 \tag{194}$$

and the velocity

$$V = V_1 = H_1\xi, \; V_2 = H_2\xi = 0$$

while p and ρ are pressure and density, respectively.

The momentum equations read

$$H_1\frac{\partial V}{\partial t} + V\left(\frac{\partial H_1}{\partial t} + \frac{\partial V}{\partial \xi}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial \xi} \tag{195}$$

and

$$-\frac{V^2}{H_1 H_2}\frac{\partial H_1}{\partial \xi} = \frac{1}{\rho}\frac{\partial p}{\partial \zeta} \; . \tag{196}$$

The continuity equation follows from the condition

$$\text{div } V = 0 \quad \text{i.e.} \quad \frac{\partial}{\partial \xi} \quad {}^{vH} \quad * \quad) \quad = \quad 0 \tag{197}$$

Equations (187)-(189) are completed by the compatibility (Lame's) equation

$$\frac{\partial}{\partial \xi}\left(\frac{1}{H_1}\frac{\partial H_2}{\partial \xi}\right) + \frac{\partial}{\partial \zeta}\left(\frac{1}{H_2}\frac{\partial H_1}{\partial \zeta}\right) = 0 \tag{198}$$

Linearizing these equations with respect to an unperturbed shear flow:

$$\check{V} = \check{V}(y) \tag{199}$$

and taking into account that for this flow the streamline coordinates coincide with the cartesian coordinates:

$$\xi = x, \; \zeta = y, \; \check{H}_1 = \check{H}_2 = 1 \; ; \tag{200}$$

one obtains after eliminating the pressure.

$$\frac{\partial^2 \breve{V}}{\partial t \partial \zeta} \frac{\partial^2 \tilde{H}_1}{\partial t \partial \zeta} + \breve{V}(\zeta) \frac{\partial^2 \breve{V}}{\partial \xi \partial \zeta} + \breve{V}^2(\zeta) \frac{\partial^2 \tilde{H}_1}{\partial \xi \partial \zeta} = 0 . \tag{201}$$

$$\frac{\partial \breve{V}}{\partial \xi} + \breve{V}(\zeta) \frac{\partial \tilde{H}_2}{\partial \xi} = 0 , \tag{202}$$

$$\frac{\partial^2 \tilde{H}_2}{\partial \xi^2} + \frac{\partial^2 \tilde{H}_1}{\partial \zeta^2} = 0 . \tag{203}$$

where $\breve{V}$, $\tilde{H}_1$, and $\tilde{H}_2$ are small perturbations of V, H_1, and H_2, respectively.

If the solution for $\breve{V}$ is assumed to be of the form

$$\breve{V} = \breve{V}(\zeta)\phi'(\zeta) \, e^{i(\alpha\xi - \beta t)} , \quad \alpha , \beta = \text{const}, \tag{204}$$

then, as follows from equations ('202) and (203),

$$\tilde{H}_2 = -\phi'(\zeta) \, e^{i(\alpha\xi - \beta t)} , \quad -\frac{\partial \tilde{H}_1}{\partial \zeta} = -\alpha^2\phi(\zeta) \, e^{i(\alpha\xi - \beta t)}. \tag{205}$$

Substituting the values (204) and (205) into equation (201) one arrives at the governing equation for $\phi(\xi)$:

$$\phi'' - \frac{c\breve{V}'(\zeta)}{\breve{V}(\zeta)[\breve{V}(\zeta) - c]}\phi' - \alpha^2\phi = 0, \quad c = \frac{\beta}{\alpha} , \tag{206}$$

which is different from the Orr-Sommerfeld equation.

If the basic flow $\breve{V}(y)$ is bounded by rigid walls:

$$Y = y_1 , \quad y = y_2 , \tag{207}$$

then the streamlines at $\eta = y_1$ and $\zeta = y_2$ must coincide with these walls, i.e.

$$\frac{\partial \tau}{\partial \xi} = -\frac{1}{H_2}\frac{\partial H_1}{\partial \zeta} \, \mathbf{n} = 0 \text{ at } y = y_1 \text{ and } y = Y_2 \tag{208}$$

in which τ and $\mathbf{n}$ are the unit tangent and the unit normal vectors to the streamlines.

Hence

$$\frac{\partial H_1}{\partial \zeta} = 0 , \quad \text{at } y = y_1 \text{ and } y = y_2 \tag{209}$$

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and therefore, with references to equations (205),

$$\phi(\zeta_1 = y_1) \quad \cdot \quad 0 \; , \quad \phi(\zeta_2 = y_2) = 0. \tag{210}$$

These equations express the boundary conditions for equations (206).

In order to show hat the stability criteria in streamline coordinates are different from those given by the Orr-Sommerfeld equations let us select a special velocity profile $\check{V}(y)$ such that the coefficient of ϕ' in equation (206) reduces to a constant. Obviously, such a profile must satisfy the first-order differential equation

$$\frac{\check{V}'}{\check{V}(V-C)} \quad - \; - \quad = y = \text{const.} \quad \text{Im} y = 0 \; ; \tag{211}$$

and consequently

$$\dot{V} = \frac{c}{1 - e^{c\gamma y}} \text{ `} \tag{212}$$

while equations (206) for this profile reduces to

$$\phi'' - c\gamma\phi' - \alpha^2\phi = 0. \tag{213}$$

Its general solution is

$$\phi = C_1 \exp(\lambda_1 y) + C_2 \exp(\lambda_2 y) \; , \tag{214}$$

where

$$\lambda_{1,2} = \frac{cy}{2} + \sqrt{\frac{c^2 y^2}{4} + \alpha^2} \tag{215}$$

Substitution of the boundary conditions (210) into equation (214) leads to a system of homogeneous equations:

$$C_1 \exp(\lambda_1 y_1) + C_2 \exp(\lambda_2 y_1 = 0 \; , \; C_1 \exp(\lambda_1 y_2) + C_2 \exp(\lambda_2 y_2) = 0 \; , \tag{216}$$

and for a non-trival solution

$$\det \begin{pmatrix} \exp(\lambda_1 y_1) & \exp(\lambda_2 y_1) \\ \exp(\lambda_1 y_2) & \exp(\lambda_2 y_2) \end{pmatrix} = \exp(\lambda_1 y_1 + \lambda_2 y_2) - \exp(\lambda_1 y_2 + \lambda_2 y_1) = 0, \tag{217}$$

i.e., $\lambda_1 = \lambda_2$; or, with reference to equation (215),

$$c = \pm i \, 2 \, \alpha/y = \pm c_0 i \; . \tag{218}$$

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Since α and γ are real, c is imaginary, and therefore. solutions (205) are unstable fOr any y_1 and y_2.

Now we will show that tile Orr-Sommerfeld equation predicts stability for the same profile. Indeed, substituting c from equation (218) into equation (217) and separating the real part of the velocity profile, one obtains

$$\mathcal{R}e\dot{V} = \pm \cot\tan (2 \alpha y) \ . \tag{219}$$

This profile has only one inflection point (at $y = \pi / (4 \alpha)$). Consequently, according to the point-of-inflection criterion proved by Tollmien, any profile of the form (219) which does include the inflection point, i.e.

$$0 \leq y_1 \leq y \leq y_2 < \frac{\pi}{4\alpha} \tag{220}$$

is stable.

It is important to emphasize that these two different results regarding the same velocity profile are not mutually exclusive: the first is related to the stability of the fluid motion referred to streamline coordinates, while the second is related to the stability of the velocity field. But which one of these approaches is actually related to the onset of turbulence? The dynamics of fluid motion, and in particular, the stability of streamlines, is directly related to the onset of turbulence inasmuch as the stability of particle trajectories is directly related to the onset of Lagrangian turbulence. At the same time, the stability of velocity fields is indirectly related to the onset of turbulence. That is why the linearized version of the classical theory of stability cannot explain the instability of plane Couette flows. In this connection it is worth noting that by an appropriate selection of α, y_1 and y_2 in equations (219) and (2'20), the velocity profile, Eq. (219), can be made as close as necessary to a straight line, thereby predicting the instability of any flow which is arbitrarily close to the Couette flow.

d.   Instability Dependence Upon **the Class of Functions**

The properties of solutions to differential equations such as existence, uniqueness and stability, have a mathematical meaning only if they are referred to a certain class of functions. For instance, as shown above in Eqs. (143) and (146), we have a unique stable solution in an open interval (156) ill the class of bounded functions, while in a

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closed interval, (157), the uniqueness and stability are not guaranteed. Most of the rt'suits concerning the properties of solutions to differential equations require differentiability (up to a certain order) of the functions describing the solutions. However, tile mathematical restrictions imposed upon the class of functions which guarantee the existence of an unique and stable solution, do not necessarily lead to the best representation of the corresponding physical phenomenon. Indeed, turning again to Eqs. (143) and (145), one notices that tile unique and stable solution (156) does not describe a cummulation effect (a snap of a whip) which is well pronounced in experiments. At the same time, an unstable solution in a closed interval (157) gives a qualitative description of this effect. Hence, pm-e mathematical restrictions imposed upon the solutions are not always consistent with the physical nature of motions. In this context, the long- term instability in classical dynamics discussed in the Section II, can be interpreted as a discrepancy between these mathematical restrictions and physical reality. This means that unpredictability in classical dynamics is a price paid for mathematical "convenience" in dealing with dynamical models. Therefore, the concept of unpredictability in dynamics should be put as unpredictability in a selected class of functions, or in a selected metrics of configuration space, or in a selected frame of reference.

Now the following problem can be posed. How to select an appropriate mathematical representation of a physical phenomena? The answer to this question will be discussed below.

2.   Dynamics In Fast Oscillating Frame of Reference

As **shown** in the previous sections, the instability, and therefore, the occurrence of chaos or turbulence in description of mechanical motions means only that these motions cannot be properly described by smooth functions if the scale of observat ions is limited. These arguments can be linked to Godel's incompleteness theorem (1931), and the Richardson's (1968) proof that the theory of elementary functions in classical analysis is undecidable.

But since instability is not an invariant of motions, the following question can be posed: is it possible to find such a new (enlarged) class of functions, or a new metric of configuration space, or a new frame of reference in order to eliminate instability'? Actually such a possibility would lead to different representative parameters describing the same

motion in such a way that small uncertainties in external forces cause small changes of these parameters. For example, in turbulent and chaotic motions, mean velocities, Reynolds Stresses, and power spectra, represent "stable" parameters, although classical governing equations neither are explicitly expressed via these parameters, nor uniquely define them.

The first step toward the enlarging of the class of functions for modeling turbulence was made by ().' Reynolds (1 895) who decomposed the velocity field into the mean and pulsating components, and actually introduced a multivalued velocity field. However, this decomposition brought new unknowns without additional governing equations, and that created a "closure" problem. In 1986 Zak has shown that the Reynolds equations can be obtained by referring the Navier-Stokes equations to a rapidly oscillating frame of reference. while the Reynolds stresses represent the contribution of inertia forces. From this viewpoint the "closure" has the saint> status as "proof" of Euclid's parallel postulate, since the motion of the frame of reference can be chosen arbitrarily. In other words, the "closure" of Reynolds equations represents a case of undecidability in classical mechanics. However, based upon the interpretation of the Reynolds stresses as inertia forces, it is reasonable to choose the motion of the frame of reference such that the inertia forces, eliminate the original instability. In other words, the enlarged class of functions should be selected such that tile solution of the original problem in that class of functions will not possess an exponential sensitivity to changes in initial conditions. This stabilization principle has been formulated and applied to chaotic and turbulent motions by Zak (1982, 1985a, 1986a,b, 1990). As shown there, the motions which are chaotic (or turbulent) in the original frame of reference can be represent ed as a sum of the "mean" motion and rapid fluctuations, while. both components are uniquely defined. It is worth emphasizing that the amplitude of velocity fluctuation is proportional to the degree of the original instability, and therefore, the rapid fluctuations can be associated with the measure of the uncertainty in the description of motion. It should be noticed that both "mean" and "fluctuation" components representing the originally chaotic motion are stable, i.e., they me not sensitive to changes of initial conditions, and are fully reproducible.

'Let us refer the original equation to motion of a non-inertial frame of reference which rapidly oscillate with respect to the original inertial frame of reference. Then the absolute velocity $\dot{q}$ can be decomposed into the relative velocity $\dot{q}_1$ and the transport velocity

$\dot{q}_2 = 2\dot{q}_{2(0)}$:

$$\dot{q} = \dot{q}_1 + 2\dot{q}_{2(0)} \cos \omega \rightarrow \infty \tag{221}$$

while $\dot{q}_1$ and $\dot{q}_2^0$ are "'slow" functions of time in the sense that

$$\omega \gg \frac{1}{\tau} \tag{222}$$

where τ is the time scale upon which the changes $\dot{q}_1$ and $\dot{q}_{2(0)}$ can be ignored.

Then for the mean $\bar{q}$:

$$q \cong q_1 \quad \text{since} \quad \int_0^{t \gg \tau} \dot{q}_{2(0)} \cos \omega t dt \simeq \frac{1}{\omega} \dot{q}_{2(0)} \sin \omega t \rightarrow 0 \text{ if } \omega \rightarrow \infty \tag{223}$$

In other words, a fast oscillating velocity practically does not change the displacements.

Taking into account that

$$\frac{\omega}{2\pi} \int_0^{2\pi/\omega} \dot{q}_1 dt \simeq \dot{q}_1 \,, \quad \int_0^{2\pi/\omega} \dot{q}_{2(0)} \sin \omega t dt = 0 \,, \text{ and } \int_0^{2\pi/\omega} \dot{q}_{2(0)}^2 \cos^2 \omega t dt = \frac{1}{2}\dot{q}_{2(0)}^2 \tag{224}$$

one can transform a system:

$$\dot{x}^i = a_j^i + b_{jm}^i x^i x^m \,, \quad i = 1, 2, \ldots n \tag{225}$$

into the following form:

$$\bar{x}_i = a_j^i \bar{x}^j + b_{jm}^i \bar{x}^j \bar{x}^m \,, + b_{jm}^i \overline{x^j x^m} \,, 2 = 1, 2, \ldots n \tag{226}$$

where $\bar{x}^i$ and $x^i x^j$ are means and [lotll~le-correlations of x^i as random variables, respectively.

As will be shown below, the transition from (225) to (226) is identical to the Reynolds transformation: i.e., being applied to the Navier-Stokes equations, it leads to the Reynolds equations, and therefore, the last terms in (226) (which is a contribution of inertial forces due to fast oscillations of the frame of reference) can be identified with the Reynolds stresses. From a mathematical viewpoint, this transformation is interpretable as enlarging the class of smooth functions to multivalued ones. Indeed, as follows from (222), for any arbitrarily small interval Δt, there always exists such a large frequency $\omega > \Delta t/2\pi$ that within this interval the velocity $\dot{q}$ runs through all its values, and actually the velocity field becomes multivalues.

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Clearly Eqs. (226) result from the time-averaging. In case of applicability of the ergodic hypothesis, t he same equations ran be obt ained from t he ensemble- averaging. However, formally the averaging procedure can be introduced axiomatically based upon the Reynolds conditions:

$$\overline{a + b} = \bar{a} + \bar{b}, \quad \overline{ka} = k\bar{a}, \quad \bar{k} = k, \quad (k = \text{const}.$$

$$\overline{\frac{\partial a}{\partial l}} = \frac{\partial \bar{a}}{\partial l}, \quad \overline{\bar{a}b} = \bar{a}\,\bar{b}$$

This leads to the identity:

$$\overline{ab} = \bar{a}\bar{b} + \overline{a'b'}$$

where $a = \bar{a} + a'$, $b = \bar{b} + b'$

Let us consider a mechanical system with N-degrees of freedom and the kinetic energy:

$$W = a_{ij}\dot{q}^i\dot{q}^j , \tag{227}$$

which q', and $\dot{q}'$ are the generalized coordinates and velocities respectively, and introduce an N-dimensional (abstract) space with the metric:

$$ds^2 = a_{sk}dq^s dq^k , \quad \dot{s}^2 = 2W. \tag{228}$$

Then the equations of motion

$$q' = q^{i(t)} \tag{229}$$

satisfy the following differential equations:

$$\ddot{q}^\alpha + \Gamma^\alpha_{\beta\gamma}\dot{q}^\beta\dot{q}^\gamma = Q^\alpha , \tag{230}$$

where Q* is the force vector, $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols:

$$\Gamma^e{}_{sk} = \frac{1}{2}\alpha^{ip}\left[\frac{\partial a_{sp}}{\partial q^k} + \frac{\partial a_{kp}}{\partial q^s} - \frac{\partial a_{sk}}{\partial q^r}\right], \quad a^{\alpha\beta}a_{\beta\gamma} = \delta^\alpha_\gamma = \begin{cases} 0 & \text{if } \alpha \neq \gamma , \\ 1 & \text{if } \alpha = \gamma . \end{cases} \tag{231}$$

Equation (230) can be interpreted as a parametrical equation of the trajectory C of a representing point M with the contravariant coordinates q^α. The unit tangent vector $T = \nu_0$ to this trajectory is defined as:

$$\tau^\alpha = \nu^\alpha_0 = \frac{dq^\alpha}{ds} = \frac{1}{\sqrt{2W}}\dot{q}^\alpha , \quad \alpha_{mn}\nu^m_0\nu^n_0 = 1, \tag{232}$$

while this unit normals $\nu_1, \nu_2, \cdots \nu_{N-1}$ are given by the Frenet equations:

$$\frac{d\nu_p^l}{ds} + \Gamma_{kq}^l \nu_p^q \frac{dq^k}{ds} = - \chi_p \nu_{p-1}^l + \chi_{p+1} \nu_{p+1}^l , \tag{233}$$

where $\chi_1, \chi_2, \cdots \nu_{N-1}$ are the curvatures of the trajectory, and S is the arc coordinate along this trajectory.

The principal normal ν_1 is coplanar with the tangent ν_0 and the force vector Q. The rest curvatures as well as the directions of the rest normals are defined by Eq. (273), Fig. 9.

For simplicity we will confine ourselves by a particular case when

$$\Gamma_{sk}^l = \text{const} \tag{234}$$

Substituting the decomposition (221) into Eqs. (230), one obtains:

$$\ddot{q}_1^\alpha + \Gamma_{\beta\delta}^\alpha \dot{q}_1^\beta \dot{q}_1^\gamma + \Gamma_{\beta\delta}^\alpha \dot{q}_{2(0)}^\beta \dot{q}_{2(0)}^\alpha = Q^\alpha \tag{235}$$

Here the terms

$$Q_{(i)}^\alpha = - \Gamma_{\beta\gamma}^\alpha \dot{q}_{2(0)}^\beta \dot{q}_{2(0)}^\gamma \tag{236}$$

represent the inertia forces caused by the transport motion of the frame of reference.

Applying the velocity decomposition similar to (221):

$$v = \bar{v} + 2\tilde{v} \cos\omega t, \quad \omega \to \infty \tag{237}$$

to the momentum equation for a continuum in Eulerian representation:

$$\rho\left(\frac{\partial \bar{v}}{\partial t} + v \nabla v\right) = \nabla \bullet \sigma \tag{238}$$

where σ is the stress tensor, one obtains:

$$\rho\left(\frac{\text{in}}{\partial t} + \bar{v} \nabla \bar{v}\right) = \nabla \bullet (\sigma + \tilde{\sigma}) \tag{239}$$

in which $\tilde{\sigma}$ is the Reynolds stress tensor with the components:

$$\tilde{\sigma}_{ij} = - \rho \overline{\tilde{v}_i \tilde{v}_j} \tag{240}$$

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In terms of the Reynolds equations. $\overline{V}$ and $\overset{\vee}{V}$ represent the mean velocity and the amplitude of fast velocity fluctuations, respectively.

The most significant advantage of the Reynolds-type equations (226), (235) and (239) is that they are explicitly expressed via the physically reproducible parameters $\overline{r^i}$, $\overline{r^i r^j}$ which describe, for instance, a mean velocity profile in turbulent motions, or a power spec - trum of chaotic attractors. However, as a price for that, these equations require a closure since the number of unknowns in there is larger than the number of equations. Actually the closure problem has existed for almost a hundred years since the Reynolds equations were derived. In the next sections, bases upon tile stabilization principle introduced by Zak, M. (1985a, 1986a,b,1990), this problem will be discussed.

3. Stabilization Principle and the Closure Problem

a. General Remarks

Revisiting the dynamical systems $(226),(235)$, and (239) which describe motions in the class of multivalued functions, one notes that these systems are not complete in the sense that the number of unknowns is larger than the number of equations. In particular, the vector which expresses the bulk contribution of the "micro-scale" motions into the averaged motion, represents excessive unknowns. Such an incompleteness creates a closure problem. This problem first was identified in connection to the Reynolds equations describing turbulent motions, The problem of turbulence arose almost a hundred years ago as a result of discrepancy between theoretical fluid mechanics and experiments. However, in spite of considerable research activity, there is no general approach to prediction of t ur- bulence based upon theoretical models. Most of the efforts were directed toward finding a " physical" law which would couple the Reynolds stresses with the rate-of-strain of t he average motion, and thereby, would represent additional equations required for the closure of the Reynolds equations. For instance, Prandtl introduced the mixing length assumption

$$\tau = \rho \ell^2 |\frac{\partial u}{\partial y}| \frac{\partial u}{\partial y} \tag{241}$$

for the t we-dimensional version of the Reynolds equation:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{1}{\rho} \frac{\partial \tau}{\partial y} , \tau = -\rho u_r v_r , \tag{242}$$

Here u, v, u_r and v_r are the mean and fluctuation velocity projections on the cartesian coordinates x and y, respectively, τ is the shear component of the Reynolds stress, and ℓ is a so-railed mixing length which is supposed to be found from experiments.

By exploiting the closure (241), Prandtl solved severa problems Of two-dimensional theory of turbulence: he found a mean velocity profile of an axi-symmetrical turbulent flow in a pipe, he described the smoothing out of velocity liscontinuity, etc., while all of his solutions were sufficiently close to experimental results.

However, tile same closure (241) failed to provide satisfactory solutions in many other cases, which means that the closure (241) can not be considered as a "physical" law. But does any "physical" law of this type (241) exist, in principle'? And is such a law necessary for the closure"! Indeed, as shown in the previous section, the Reynolds stresses can be interpreted as a contribution of the inertia forces of a rapidly oscillating frame of reference. while this frame of reference can be chosen [arbitrarily! However, such an interpretation leads to another question: is it possible to find such a frame of reference which provides the "best" representation of the motion"? Obviously, in this representation the motion must be stable, and therefore, the restoration of stability of the originally unstable motion can be chosen as the main criterion for selection of tile frame of reference, and therefore, of the Reynolds stresses. From the mathematical viewpoint, it means that if the original motion is unstable in the clam of smooth functions, this instability can be eliminated by enlarging the class of functions. From that viewpoint, the Prandtl's closure (241) can be treated as a feedback which stabilizes an originally unstable laminar flow, Indeed, turning, for instance, to a plane Poisson flow with the parabolic velocity profile, one arrives to its instability if the Reynolds number is larger than $R_{cr} \cong 5772$. Experiments show that a new steady turbulent profile is not parabolic any more: it is very flat near the center and is very steep near the walls. The same profile follows from the Prandtl solution based upon the closure (241). But since this profile can be experimentally observed, it must be stable, and this stabilization is carried out by the "feedback" (241).

b.   Formulation of the Stabilization Principle

Based upon remarks made in the previous section, we will now formulate the following stabilization principle. Consider a dynamical model which in some domain of its parameters becomes unstable in the class of differentiable functions, i.e., its instability leads to

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an unbounded growth of ignorable variables. As noticed earlier, this means that the corresponding physical phenomena cannot be adequately described in the class of differentiable functions, and the original model must be modified. The modification of the model should be based upon the enlarging the original class of functions in such a way that tile instability is eliminated. This mathematical formulation can be complemented and specified by the following physical reasonings: The application of the Reynolds-averaging conditions to any nonlinear dynamical model leads to another nonlinear system which differs from the original one by additional variables - the Reynolds "stress" (see Eqs (226), (235) and (239)). In a symbolic form, the transformation from the Newtonian (Y.) to the Reynolds (R.) dynamics can be presented as:

$$R \bullet x = N \bullet x + \sigma_R$$

If the original dynamical system

$$N \bullet x = 0$$

is unstable, but the Reynolds-averaged system

$$N \bullet x + \sigma_R = 0$$

is Stable, obviously, the stabilization is performed by the Reynolds stresses σ_R: driven by the mechanism of instability of the original model, they grow until the instability is suppressed down to a neutral stability. As will be shown below, the last condition uniquely defines σ_R as well as all the averaged parameters of the dynamical system. Mathematical justification of the neutral stability of Reynolds-averaged models will be given in the section 4,C (see Eqs. (275) and (276).

Experimental verification of neutral stability of free turbulent jets was reported by Lessen, M. (1976),

In the next sections the stabilization principle will be applied to prediction of postinstability behavior of fluids (turbulent motions), and of finite-dimensional dynamical systems (chaos).

4. **Application** of the Stabilization Principle to Predictions of Chaotic Motions

The strategy for application of the stabilization principle to predict chaotic motions for inertial, potential and dissipative systems will be presented in this section.

a. **inertial** Motions

In order to clarify the main idea of the approach, let us turn to the inertial motion of a particle M of unit mass in a smooth pseudosphere S having a constant negative curvature (15). As shown thin, the orbital instability, and therefore, the chaotic behavior of the particle M can be eliminated by the elastic force (40):

$$F = -\alpha^2 \epsilon \ , \alpha^2 = \text{const.} \ > \ -2WG, G \ < \ 0 \tag{243}$$

proportional to the normal deviation ϵ from the geodesic trajectory which is applied to the particle M. But such a force can appear as an inertial force if the motion of the particle M is referred to an appropriate non-inertial system of coordinates.

Indeed, so far this motion was referred to an inertial system of coordinates q_1, q_2, whew q_1 is the coordinate along the geodesic meridians, and q_2 is the coordinate along the parallels. Let us introduce now a frame of reference which rotates about the axis of symmetry of the pseudoshere with the rapidly oscillating transport velocity:

$$\dot{\epsilon} = 2\dot{\epsilon}_0 \ \cos \omega t, \ \ \omega \to \infty \tag{244}$$

so that the components of the resultant velocity along the meridians and parallels are, respectively:

$$v_1 \ = \ \dot{q}_1, \ \ v_2 \ = \ \dot{q}_2 + 2\dot{\epsilon}_0 \ \cos \omega t \tag{245}$$

Since Eq. (245) has the same structure as Eq. (221), the Lagrangian of the motion of the particle M relative to the new (non-inertial) frame of reference can be written in the following form:(see Eq. 2'2):

$$L^* = \dot{q}_1^2 \ - \ \frac{1}{G_0}e^{-2\sqrt{-G_0}q_1} \ (\dot{q}_2^2 + \dot{\epsilon}_0^2) \tag{246}$$

The last term in Eq. (246) represents the contribution of the inertia forces in the new frame of reference.

So far the transport velocity $\dot{\epsilon}_0$ was not specified, and therefore, the Lagrangion (252) has the same element of arbitrariness as the governing equations (235) describing chaotic motions. Now, based upon the stabilization principle, we are going to specify the transport motion in such a way that the original orbital instability of the inertial motion of the particle M is eliminated. Turning to the condition (42), one obtains:

$$\frac{\partial^2 L}{\partial \epsilon^2} \geq -2WG_0 \tag{247}$$

where $W = \frac{1}{2}mv_0^2$ is the kinetic energy of the particle. This condition can be satisfied if the transport velocity $\dot{\epsilon}_0$ is coupled with the normal deviation ϵ as following:

$$-\frac{1}{G_0}\epsilon^{-2\sqrt{-G_0}q_1}\dot{\epsilon}_0^2 = -WG_0\epsilon^2 \tag{248}$$

As follows from Eq. (45), in this limit case the Lyapunov exponent of the relative motion in the new (non-inertial) frame of reference will be zero:

$$\sigma = \sqrt{-G_0 - \frac{\alpha^2}{W}} = 0 \ , \ \ \alpha^2 = \frac{\partial^2 L}{\partial \epsilon^2} \tag{249}$$

and the trajectories of perturbed motions do not diverge. The normal deviation from the trajectory of the relative motion (in case of zero perturbed velocity $\dot{\epsilon}_0$) can be written in the following form:

$$\epsilon = \epsilon_0 = \text{Const} \ , \ \epsilon_0 = \epsilon(t = 0) \tag{250}$$

which means that in the new frame of reference an initial error ϵ_0 does not grow - it remains constant. The relative motion along the trajectory is described by the differential equation following from the Lagrangion (246) which takes the following form (after substituting Eq. (248).

$$L = \dot{q}_1^2 - \frac{1}{G_0}\epsilon^{-2\sqrt{-G_0}q_1}\dot{q}_2 \ - WG_0\epsilon^2 \tag{251}$$

i.e.,

$$\ddot{q}_1 - \frac{2\sqrt{-G_0}}{G_0}\epsilon^{-2\sqrt{-G_0}q_1}\dot{q}_2 = 0 \tag{252}$$

But the original (unperturbed) motion was directed along **the meridians, i.e.,** $\dot{q}_2 \equiv 0$. Consequently,

$$\ddot{q}_1 = 0, \ \ \dot{q}_1 = \mathbf{u}^{"} = \text{Const} \tag{253}$$

i.e., the relative motion along the trajectory remains unchanged.

Returning to the original (inertial) system one obtains the resultant velocity by summing the relative and transport velocities:

$$v_\tau = v_0, \tag{254}$$

$$v_t = -mv_0^2 G_0 \epsilon_0 \cos \omega t \quad (\omega \to \infty), \tag{255}$$

in which v_τ and v_t are the velocity components parallel and normal to the undisturbed (geodesic) trajectory, respectively.

The equations of the disturbed motion in the original frame of reference are

$$\sigma = v_0 t, \tag{256}$$

$$\epsilon = \epsilon_0 + (\frac{1}{\omega} mv_0^2 G_0 \epsilon_0 \sin \omega t) \quad (\omega \to \infty), \tag{257}$$

in which σ is the coordinate along the undisturbed (geodesic) trajectory.

As follows from Eqs. (254-257) the motion in the original frame of reference is stable in the sense t h a t the current deviations of displacements and velocities do not exceed their initial values. However, the displacement - time function (257) is not differentiable because its derivative (255) is multivalued. Indeed, for any arbitrarily small interval Δt there always exists such a large frequency $\omega > \Delta t / 2\pi$ that within this interval the velocity (255) runs through all its values. In other words, one arrived at stability in the class of non-differentiable functions. (The mathematical meaning of this result will be discussed MOW.)

Thus, chaotic motion of a particle on a smooth pseudosphere is represented by the "mean" motion (256) along the undisturbed geodesic trajectory (with the constant velocit y (2 4 4)) and the fluctuation motion (257) normal to this trajectory. The "amplitude" of these fluctuations is vanishingly small, but the velocity "amplitude" is finite. It is worth emphasizing that this amplitude is proportional to the gaussian curvature of the surface S, it?., to the degree of the orbital instability. Therefore, it can be associated with the measure of the uncertainty in the description of the motion.

It is worth mentioning that both "mean" and "fluctuation" components representing the originally chaotic motion are stable. That is why they are not sensitive to initial uncertainties and are fully reproducible. In other words, s u c h a representation of the originally chaotic motion is deterministic.

One should notice that the condition $\omega \to \infty$ is a mathematical idealization. Practically, ω is finite:

$$\omega \gg 1/T, \tag{258}$$

where T is a time scale over which changes of the parameters of the motion are negligible. In the same sense the concepts of differentiability and multivaluedness have to be understood. Indeed, the multivaluedness of the functions (261) and (262) means that the time interval between two different values of these functions is smaller than the scale of observation T Of the examined motion, and therefore, these values can be associated with "almost" the same argument.

As discussed above, the concept of stability is related to a certain class of functions, or a type of space: the same solution can be stable in one space and unstable in another, depending on the definition of the "distance" between two solutions. Indeed, if the distance between the solutions in (263) is defined as

$$\rho = \sum_{k=0}^{n} \max |\epsilon_2^{(k)}(t) - \epsilon_1^{(k)}(t)|, \tag{259}$$

then the solution (257) is stable for $n = 0, 1$, but it is unstable for $n = 2, 3, \ldots$ since its derivatives $\epsilon^{(2)}, \epsilon^{(3)}, \ldots$ etc., are unbounded. In other words, the concept of stability as well as chaos is an attribute of a mathematical model rather than of a physical phenomenon.

Hence, from a formal mathematical point of view, the occurrence of chaos in description of mechanical motions means only that these motions cannot be properly described by smooth functions if the scale of observation is finite.

One can notice that the application of the stabilization principle to representation of chaotic motions in Lagrangion dynamics can be linked to a control problem. Indeed, we are introducing additional rapidly fluctuating forces (coming from non-inertial motions of the frame of reference) which are coupled with the parameters of motions in such a way that the original instability is eliminated.

In the particular case of an inertial motion of a particle M on a pseudosphere, the rate of divergence of the trajectories was constant (see Eq. (15)) which means that local and global Lyapunov exponents are the same. That is why by t'laminating the positive local Lyapunov exponent we "automatically" eliminate the global one. In the general case, the

situation is more complex: the local Lyapunov exponents depend upon the position of the system, and by eliminating all the local positive Lyapunov exponents, one over stabilizes the motion. Indeed, non-positive global Lyapunov exponents can exist even if the local ones are positive in some domain of space where the motion can occur. As we will see later, the elimination of global Lyapunov experiments is a much harder problem, and that is why in many practical situations, we will confine ourselves with the easier problem of elimination of local exponents i.e., with the over stabilized representations.

b.   **Potential** Motions

Based upon Eqs. (235), for potential motions, the governing equations can be writ en in the following form:

$$\ddot{q}^\alpha + \Gamma^\alpha_{\beta\delta}\dot{q}^\beta\dot{q}^\delta = -\frac{\partial\Pi}{\partial q^\alpha} + a^\alpha_{(i)} \tag{260}$$

$$\frac{\partial\Pi}{\partial q^\alpha} = -Q^\alpha \tag{261}$$

where Π is the potential energy of the dynamical system, and $Q^\alpha_{(i)}$ are the inertia forces (or the "Reynolds stresses" caused by the rapidly oscillating transport motion of the frame of reference (see Eqs. (236)).

For simplicity, we will confine ourselves by a two-dimensional dynamical system assuming that $\alpha = 1, 2$.

Following the same strategy as those applied to inertial motions, let us couple tilt' inertia forces with the parameters of the dynamical system in such a way that the original orbital instability (if it occurs) is eliminated. For that purpose, first we will represent this forces in the form:

$$a^\alpha_{(i)} = -\frac{\partial\Pi_{(i)}}{\partial q^\alpha} \tag{262}$$

where $\Pi_{(i)}$ is a fictitious potential energy equivalent to the kinetic energy of the fluctuations. Thin, turning to the criteria of local orbital stability (38), one finds this potential energy $\Pi_{(i)}$, and consequently, the inertia forces $a^\alpha_{(i)}$ from the condition that original local orbital instability is eliminated:

$$G + 3\left[\frac{\nabla(\Pi + \Pi_{(i)}) \bullet \mathbf{n}}{2W}\right]^2 + \frac{1}{2W}\left|\frac{\partial^2(\Pi + \Pi_{(i)})}{\partial q^i \partial q^j} - \Gamma^k_{ij}\frac{\partial(\Pi}{\partial q^k}\mathbf{n} + \Pi_{(i)}\right|n^i n^j = 0 \quad i,j = 1,2. \tag{263}$$

Here W, G, and Γ_{ij}^k are defined by the parameters of the dynamical system (260) via Eqs. (27), (29) and (30), respectively, and n_i are the contravariant components of the unit normal n to the trajectory of the basic function.

Eq. (263) contains only one unknown $\Pi_{(i)}$ which can be found from it, and that will define the inertia forces, or the "Reynold stresses" (262)

It should be noticed that unlike the case of the inertial motion of a particle on a pseudosphere, here the Goeussian curvature G, as well as the gradients of the potential energy Π, are not constants, and consequently, the local Lyapunov exponents may be different from the global ones. This means that the condition (263) eliminates local positive exponents, and therefore, the solution to Eqs. (260) and (263) represents an over stabilized motion. Obviously, elimination of only global positive Lyapunov exponents would lead to solutions with less uncertainties while some of local exponents in certain domains of the phase space may even remain positive. However, the strategy for elimination of global positive exponents is more sophisticated, and it can be implemented only numerically.

It is worth noting that Eq. (269) is simplified to the following:

$$G + \frac{1}{2W}\left[\frac{\partial^2(\Pi + \Pi_{(i)})}{\partial q^i \partial q^j}\right] n^i n^j = 0, \tag{264}$$

if the basic motion is characterized by zero potential forces

$$\frac{\partial \Pi}{\partial q^i} = 0 \tag{265}$$

It may occur, for instance, when the dynamical system is in a relative equilibrium with respect to a moving frame.

Examples Of application of the stabilization principle to elastic systems and to ideal fluid are given by Zak, M. (1987, and 1986a, respectively).

c* **General Case**

, When motions of a dynamical system are not potential, in many cases it is more convenient to represent Eqs. (235) in the form of a system of first order differential equations. For simplicity, we confine ourselves by dynamical systems which can be presented in the following form:

$$\dot{x}^i = a_j^i x^j + b_{jm}^i x^j x^m, \quad i = 1, 2, \dots, n \tag{266}$$

Applying the transformation (221) to the variables x^i, one arrives into the following Reynolds-type equations (which are equivalent to Eqs. (235)):

$$\dot{\bar{x}}^i = a^i_j \bar{x}^j + b^i_{jm} \bar{x}^j \bar{x}^m + b^i_{jm} \overline{x^j x^m}, \quad i = 1, 2, \ldots, n, \tag{267}$$

with the additional terms $b^i_{jm} \overline{x^j x^m}$ representing the Reynolds stresses.

a.   The Closure Problem

As in the previous cases, because of additional unknowns $\overline{x^j x^m}$ in (267), the closure problem arises. Analogously, we will seek additional coupling between the mean motion and the fluctuations:

$$\overline{x^j x^m} = a^{jm}_l \bar{x}^l \bar{x}^l + a^{jm}_{ln} \bar{x}^l \bar{x}^n \ldots \text{etc.,} \tag{268}$$

based upon the stabilization principle the application of which will be clarified below.

Firstly, we recall that the solutions to equations (266) are chaotic, and consequently, some of the Lyapunov exponents of equation (266) are positive:

$$\lambda^+_m > 0, m = 1, 2, \quad . \quad . \quad ..s. \tag{269}$$

Secondly, we are looking for a decomposition in which the mean motion is periodic, rather than chaotic. Hence, the fluctuations should be coupled with the mean motion such that all positive Lyapunov exponents become zero, while the rest of the expondents are unchanged. Indeed, in this case the mean motion is a regular motion which is the "closest" to the original chaotic motion. Since the Lyapunov exponents for the system (267, 268) depend on the "feedback" coefficients a^j_l ')', a^{jm}_{ln} etc., the closure can now be formulated as follows:

$$\lambda^+_i (a^{jm}_l, a^{jm}_{ln}, \ldots) = \text{o}, \quad i = 1, 2, \ldots, S_+,$$
$$\lambda^0_i (a^{jm}_l, a^{jm}_{ln}, \ldots) = \lambda^0_i (0, 0, \ldots), \quad i = 1, 2, \ldots \text{ s0},$$
$$\lambda^-_i (a^{jm}_l, a^{jm}_{ln}, \ldots) = \lambda^-_i (0, 0, \ldots), \quad \textbf{i = 1, 2, ..., S-.,} \tag{270}$$

in which λ^+, λ^0 and λ^- are positive, zero and negative Lyapunov exponents, respectively. Obviously, those coefficients a^{jm}_l which do not appear in equation (270), must be zero.

Thus, the system (267, 26S, 270) is closed. It defines the regular mean motion and fluctuations which represent the original chaotic motion. Since all the Lyapunov exponents for this systt'in are not positive, the solution is stable and predictable in the sense that small changes in the initial conditions cause small changes in both the mean motion and the fluctuations.

In the next subsection the application of this approach to the Lorenz strange attractor is illustrated.

b.   Higher **Order** Approximations

The Reynolds decomposition of the variables x^i in Eq. (266) generates not only pair correlations $\overline{x^i x^i}$, but also correlations of higher orders, such as triple correlations $\overline{x^i x^j x^k}$, quadruple correlations $\overline{x^i x^j x^k x^m}$ et c

Indeed, multiplying equations (266) by x^k and averaging and combining the results, one obtain the governing equations for the pair correlations $\overline{x^i\,x^k}$:

$$\overline{\dot{x^i x^k}} = a^i_j\overline{x^j x^k} + a^k_j\overline{x^j x^i} + b^i_{jm}(\overline{x^k x^j x^m} + \overline{x^k x^j}\ \overline{x^m} + \overline{x^k x^m}\ \overline{x^j})$$

$$+ b_{jm}(\overline{x^i x^j x^m} + \overline{x^i x^j}\ \overline{x^m} + \overline{x^i x^m}\ \overline{x^j}), \tag{271}$$

which contain nine additional triple correlations $\overline{x^i x^j x^k}$.

Similar equations for the triple correlations will contain all the quadruple correlations etc. In general, one arrives at an infinite hierarchy of equations which are open, since any first N equations relate (N + 1) correlations.

From this viewpoint all the closures discussed above can be considered as first-order approximations w h i c h defined only the mean components of the chaotic motions. In order to define both the mean motion and the double correlations, one should consider the Reynolds equations (267) together with equations (271), In this case the evolution of the double correlations is already prescribed by equations (271), and consequently, the stabilizing feedback must now couple the triple correlations with the mean and pair correlation Component s:

$$\overline{x^i x^j x^m} = F(\overline{x^k}, \overline{x^l x^m}). \tag{272}$$

The system (267, 271, 272) will define periodic mean and pair correlation components. It is **possible** that the mean components may be different from those found before (in the saint: way in which the second-order approximation may be different from tilt' first-order one).

The higher order approximations can be introduced using the same procedure.

c.   Computational Strategy

As follows" from the above, tilt' closure, i.e., the stabilizing feedback between the Reynolds stresses and the mean components of the motion, can be written in the explicit form only if the criteria for the onset of chaos are formulated explicitly. Since such a situation is an exception rather than a rule, we develop below a computational strategy which allows one to find the closure regardless of the complexity of the original equations.

We will demonstate this strategy using equations (266). The same strategy will be suitable for the Navier-Stokes equations, since after an appropriate discretization technique they reduce to tilt' form (266).

Turning to equation (267), which follow from equations (266), as a result of the Reynolds decomposition, let us linearize them with respect to the original ("laminar") state of x_0^i:

$$\dot{\bar{x}} = (a_j^i + 2b_{jm}^i \bar{x}_0^m)\bar{x}^j, \quad \text{with } \overline{\bar{x}^i \bar{x}^j} = 0 \text{ at } \bar{x}^i = \bar{x}_0^i. \tag{273}$$

Introducing small "laminar" disturbances in the form

$$\bar{x}^i = \bar{x}_*^i \exp(\lambda_0 t), \tag{274}$$

one arrives at a truncated analog of the Orr-Sommerfeld equations:

$$\lambda^0 \delta_j^i = (a_j^i + 2b_{jm}^i \bar{x}_0^m)\bar{x}^j, \tag{275}$$

where the local eigenvalues of equations (267)

$$\lambda^0 = \lambda_1^0, \lambda_2^0, \dots, \lambda_n^0, \tag{276}$$

are the roots of the characteristic equation

$$\det(\lambda^0 \delta_j^i - a_j^i - 2b_{jm}^i \bar{x}_0^m) = 0 \tag{277}$$

Applying the same procedure to the second-order Reynolds equations (271), one obtains, instead of equation (272)

$$\frac{1}{2}\lambda^0 \delta_k^i = (a_j^i + 2b_{jm}^i \bar{x}_0^m)\overline{\bar{x}^j \bar{x}^k} \tag{278}$$

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and, therefore, the local eigenvalues of equations (271 ") are twice as large as those for equations (267), i.e., instead of equation (274):

$$\overline{x^i x^k} = \sim \exp 2\lambda_0 t).$$ (279)

If the original "laminar" state $\overline{x}_0^i$ is unstable, i.e., there are λ_i^0 with positive real parts in equation (276),

$$\Re e \lambda_i^0 > 0,$$ (280)

then the pair correlations (279) will grow much faster than the mean motion disturbances (277), and one can assume that these correlations will be large enough to stabilize equations (267) while the mean motion will remain sufficiently close to its original state $\overline{x}_0^i$. This property makes possible the following computational strategy.

Let us seek a closure to equations (267) in the neighborhood of the original laminar state $\overline{x}_0^i$ in the form

$$b_m^i \overline{x^j x^m} = C_m^i \overline{x}^j ,$$ (281)

Substituting equation (281) into equation (267) and linearizing them with respect to the original "laminar" state $\overline{x}_0^i$, one obtains

$$\overline{x}^i = (a_j^i + 2b_{jm}^i \overline{x}_0^m + C_j^i)\overline{x}^j,$$ (282)

While the eigenvalues for this equation follows from

$$\det(\lambda \delta_j^i - a_j^i - 2b_{jm}^i \overline{x}_0^m - C_j^i) = 0.$$ (283)

The sought coefficients C_j^i must be selected such that

$$\Re e \lambda_i = \frac{1}{2}(\Re e \check{\lambda}_i - |(\Re e \check{\lambda}_i|).$$ (284)

Indeed, in this case all the positive real parts of the local eigenvalues causing the instability of the "laminar flow" become zero, while the rest of these eigenvalues remain unchanged.

In order to find C_j^i from condition (284), we diagonalize the matrix

$$a_j^i + 2b_{jm}^i \overline{x}_0^m = \{F_{ij}\}$$ (285)

such that

$$\Theta^{-1} F \Theta = [\lambda_i, \ldots, \lambda_n].$$ (286)

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Then the matrix of the sought coefficients

$$C_j^n = \{C_{ij}\} \tag{287}$$

is found to be

$$\{C_{ij}\} = \frac{1}{2}\Theta[C_1, C_2, \ldots C_n]\Theta^{-1}, \tag{288}$$

in which

$$C_i = -\frac{1}{2}(\Re e \lambda_i + |(\Re e \lambda_i|). \tag{289}$$

Substituting equation (248) into equation (282), one obtains a linearized governing equation for the turbulent or chaotic motion at the very beginning of the transition from the laminar motion. Selecting a small time step $\triangle t_1$, one can find the next state $\bar{x}_1^i$:

$$\bar{x}_1^i = \bar{x}_0 + \dot{\bar{x}}_0 \triangle t_1. \tag{290}$$

Repeating this procedure for x_1^i, $\triangle t_2$, x_2^i, $\triangle t$ etc., one arrives at the evolution of the turbulence, or Chaos. The process ends when the solution approaches a regular (static or periodic) attractor whose existence is assumed.

A numerical implementation of this strategy can be based upon a direct suppression of exponential growths of errors in initial conditions by means of an appropriate selection of the Reynolds stresses. As an example of application of such a strategy, we will illustrate prediction of the probabilistic structure of the Lorenz attractor by using the stabilization principle.

Applying the Reynolds transformation to the Lorenz attractor:

$$\dot{x} = -\sigma x + \sigma y$$
$$\dot{y} = -xz + rx - y \tag{291}$$
$$\dot{z} = xy - bz$$

one obtains

$$\dot{\bar{x}} = -\sigma\bar{x} + \sigma\bar{y}$$
$$\dot{\bar{y}} = r\bar{x} - \bar{y} - \bar{x}\,\bar{z} - \overline{xz} \tag{292}$$
$$\dot{\bar{z}} = -b\bar{z} + \bar{x}\,\bar{y} + \overline{xy}$$

where $\bar{x}, \bar{y}, \bar{z}$ are the mean values of $x, y,$ and z, while $\overline{xz}$ and $\overline{xy}$ are double correlations representing the Reynolds "stresses".

As extra-variables, tilt'st> double correlations must be found from the condition that they suppress the positive Lyapunov exponent down to zero. In this case, both the mean and the double- correlations components of the motion will be represented by periodic attractors, i.e., in a fully deterministic way.

Numerical implementation of this strategy performed for $\sigma = 10$, r = 28, and b = 8/3 leads to the following results. Fig. 1 0 represents the original chaotic attractor as a solution to Eq. (291). In Fig. 11, this attractor is decomposed into two deterministic (periodic) motions: the mean motion (Fig. 1 1a) and the double correlations, it'., the Reynold's stresses (Fig. 1 1b, c). In order to find all the double correlations, one should exploit the system for triple- correlations which can be obtained in a straight- forward way from Eq. (291). In this sytem all the triple-correlations, as extra-variables, must be found from stabilization principle in a similar way. By continuing this process, one can find the probabilistic structure of the solution to the Lorenz equations (291) to a required accuracy.

It should be stressed that the solution to Eqs. (292) plotted in Fig. 11 is stable (in the new class of functions which includes "multivalued" fluctuations): small changes in initial conditions will lead to small changes in the solution.

One should recall that although equations (292) are different from the original Lorenz equations (291), they describe the same physical phenomenon in a specially selected fast oscillating frame of reference.

ACKNOWLEDGEMENT

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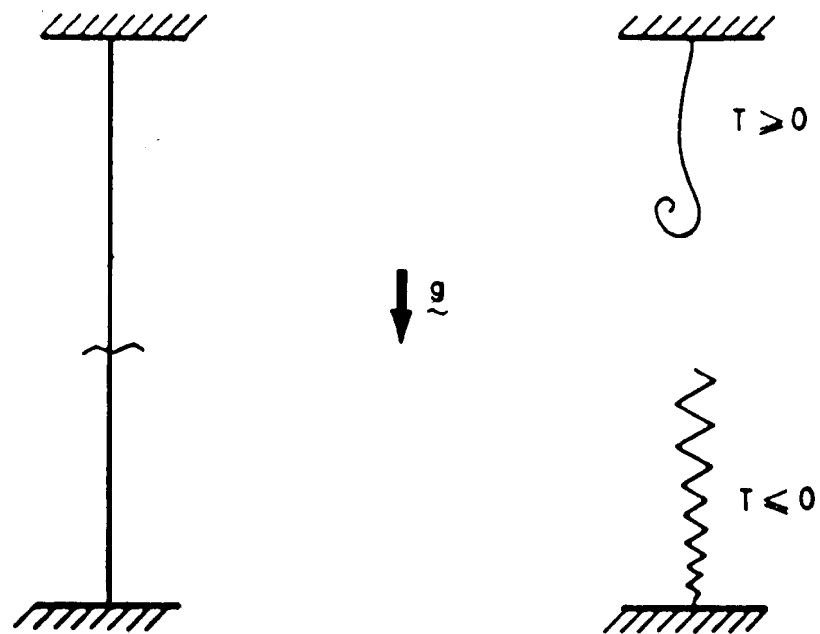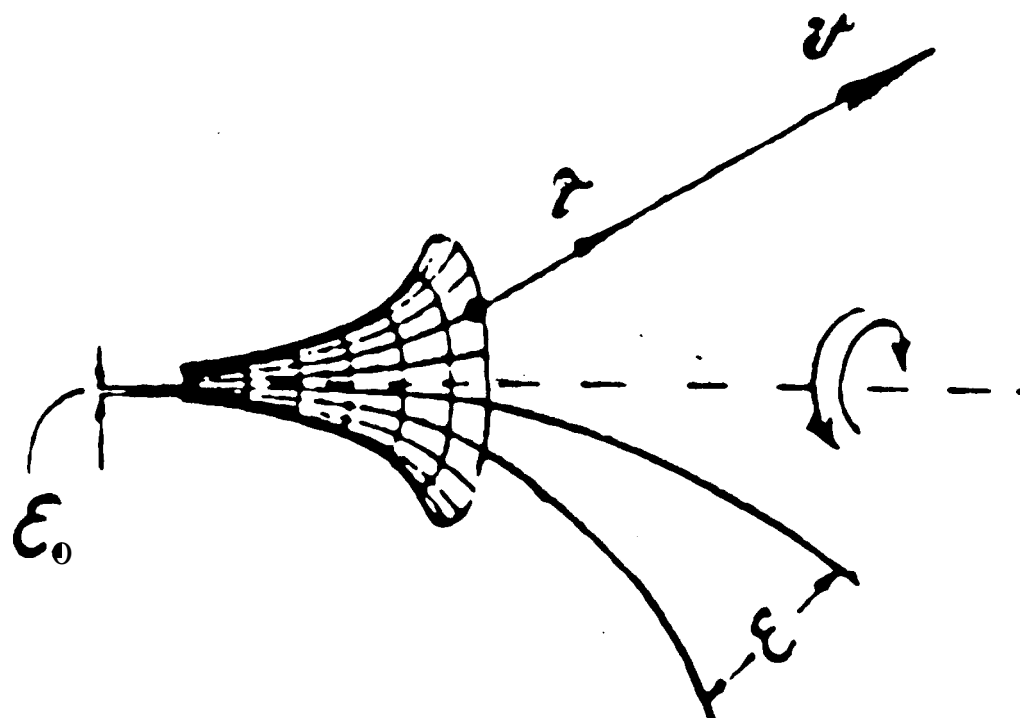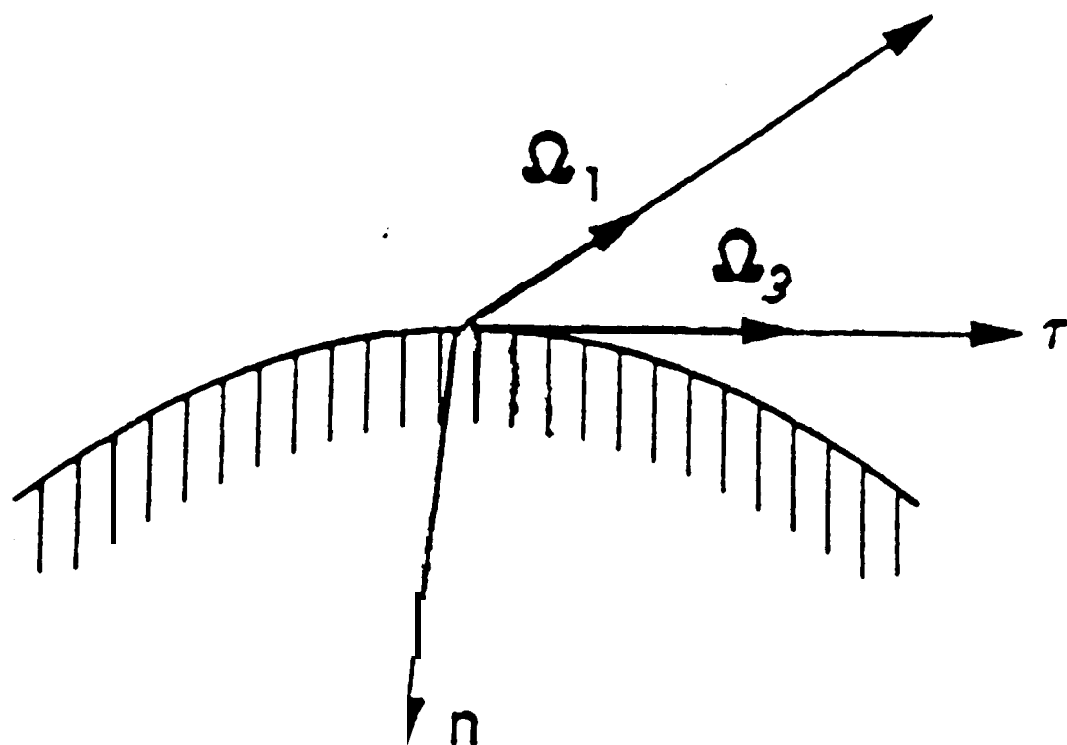
Figure L

Figure 2

Figure 3
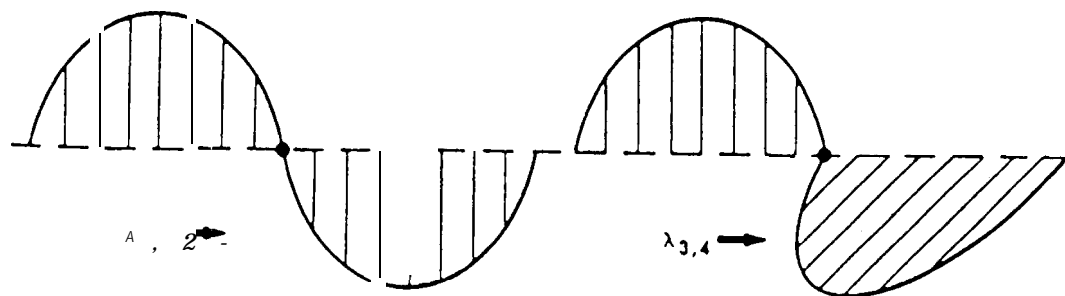
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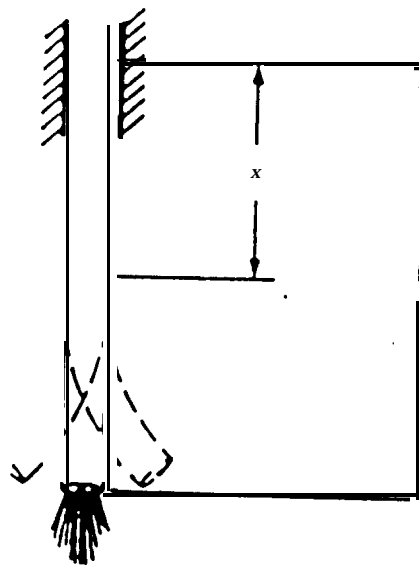
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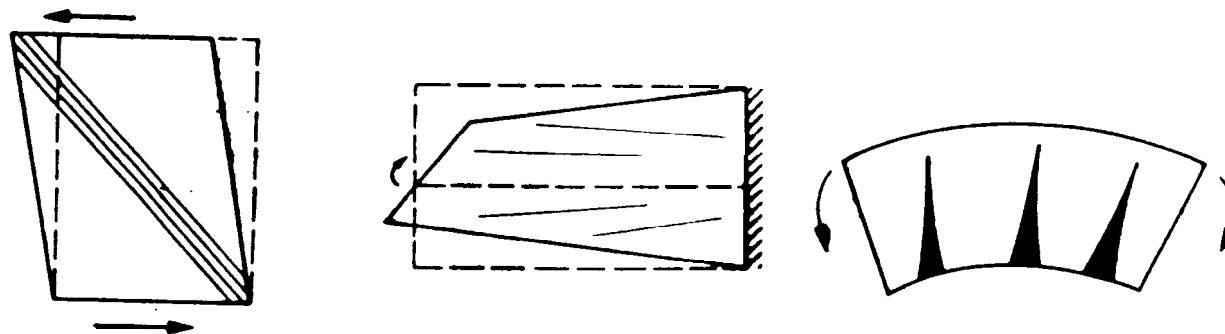
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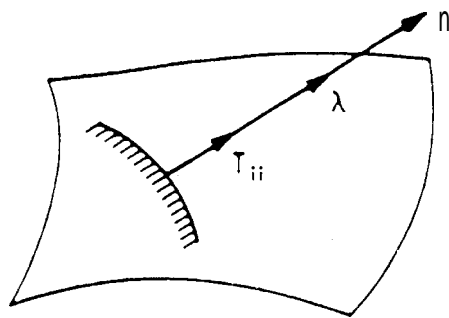
(a)

$G < 0$
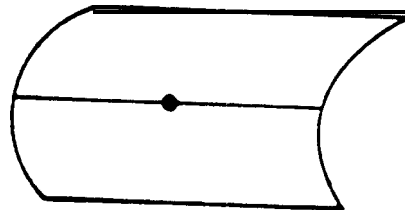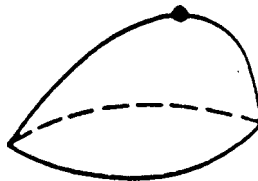
(b)

$G = 0$

(c)
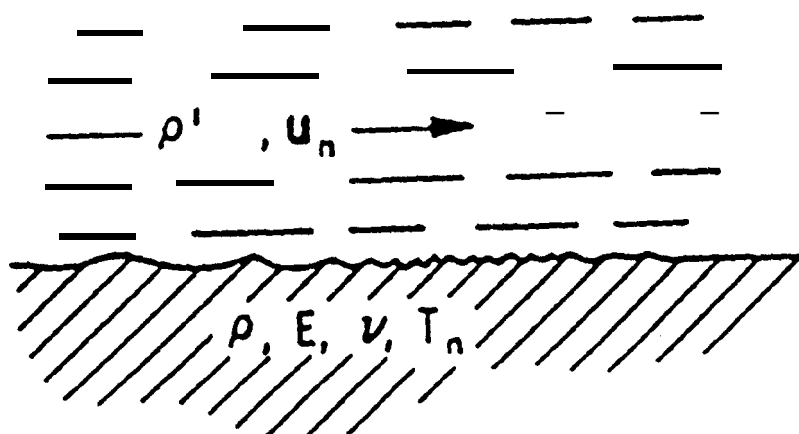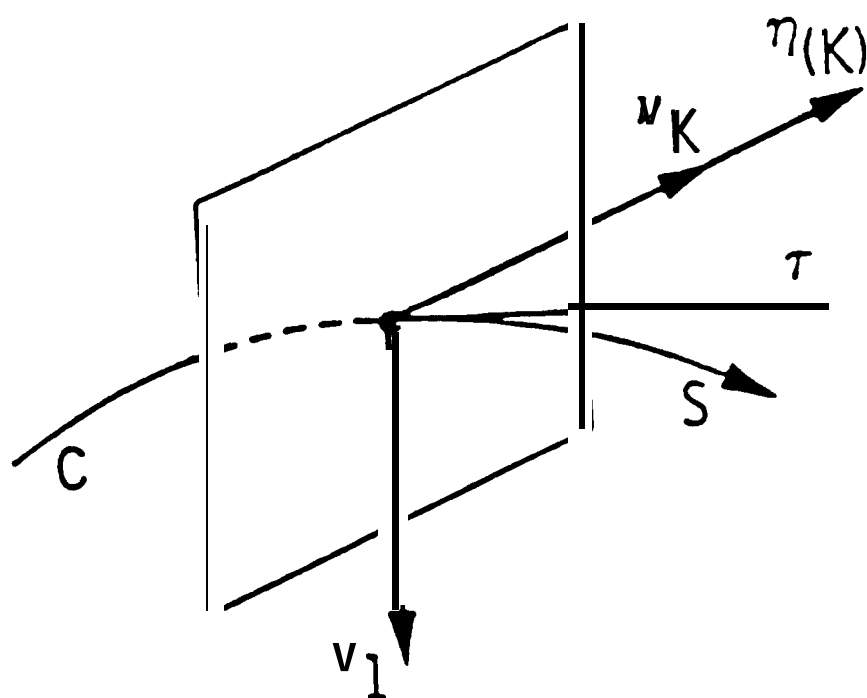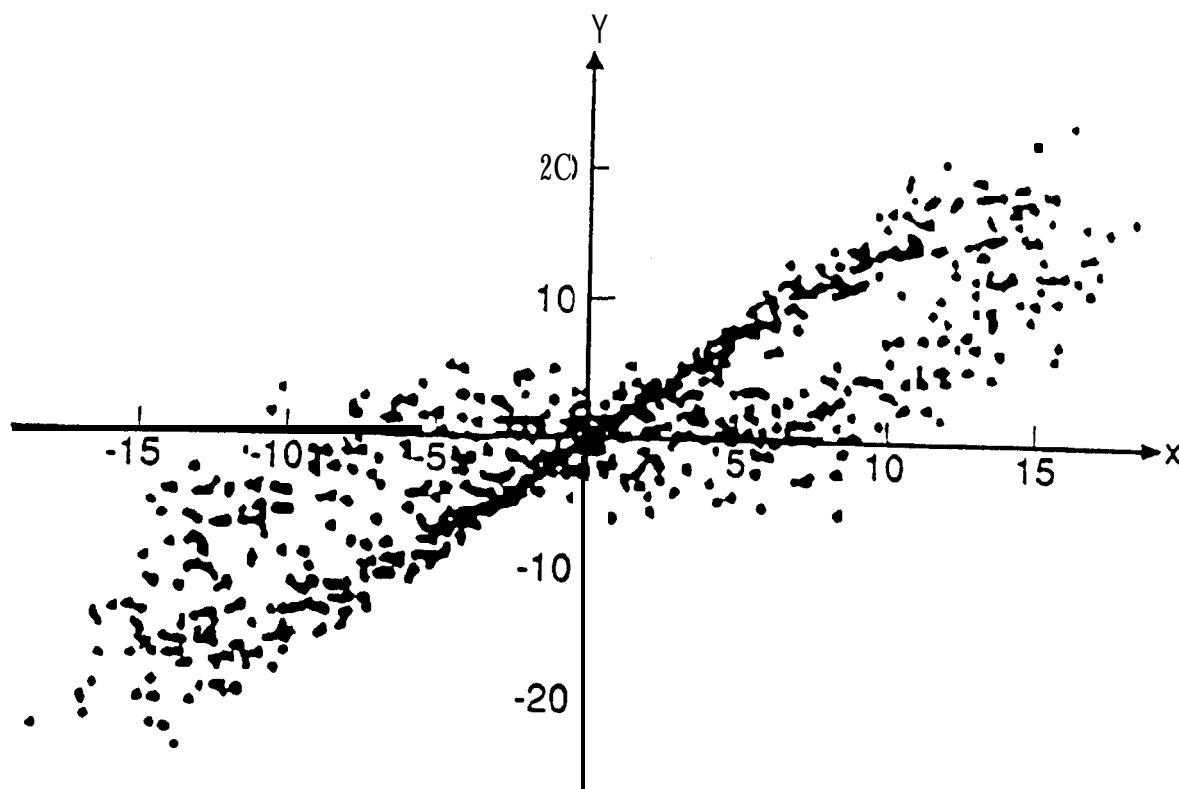
$G > 0$

(d)

Figure 7

Figure 8

Figure 9

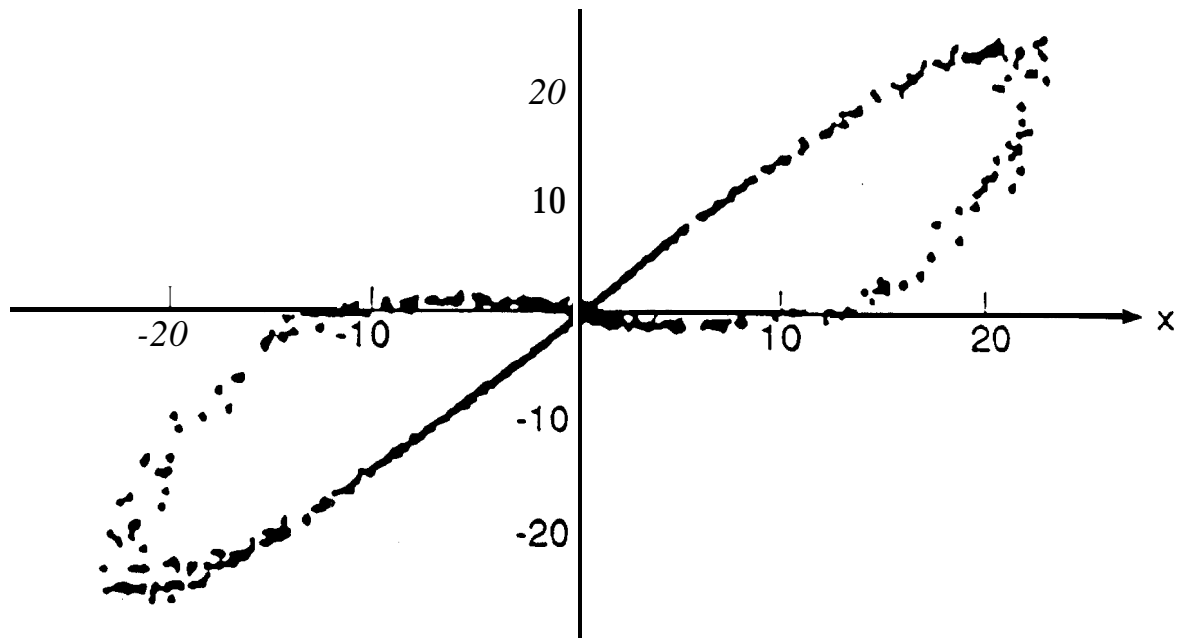Figure 10, x vs. y plot for 1,000,000 points sampled at 1000 points.

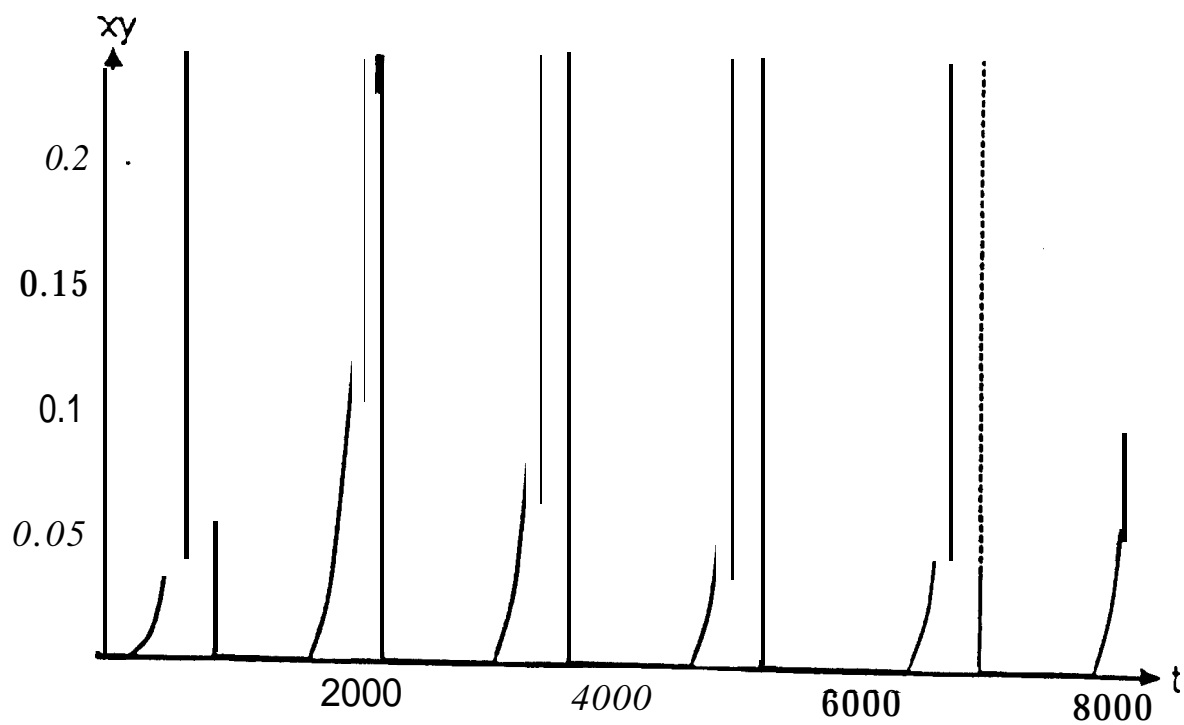Figure 11a. x vs. Y plot for 1,000,000 points sampled at 1000 points.

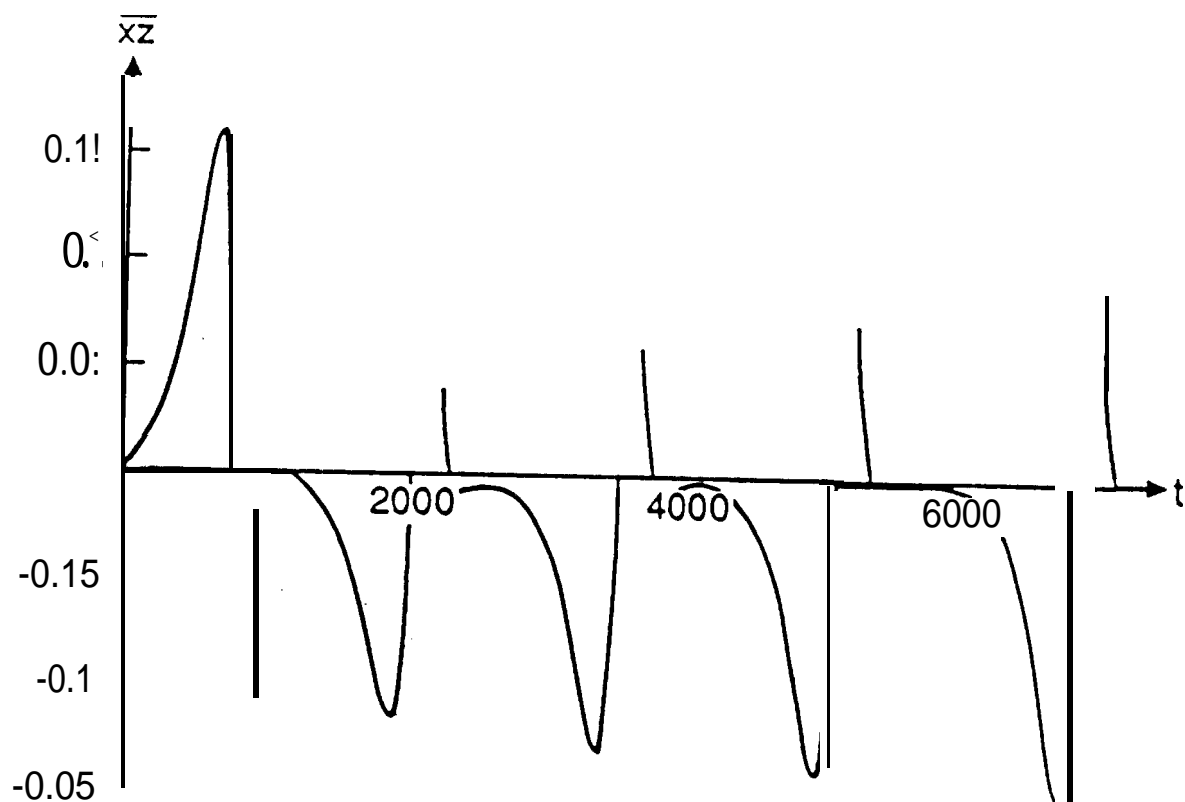Figure 11b. $\overline{xy}$: Double correlations. Plot over time (8000 points).

Figure 11c. $\overline{xz}$: Double correlations. Plot over time (8000 points).